# Lecture 1 Motivation for complex analysis

## 1.i) Introduction + Syllabus

## 1.ii) Motivation for Complex Analysis

Complex analysis studies complex-differentiable functions $f: \mathbb{C} \to \mathbb{C}$.

Ex 1.1: This condition is Much stronger than real differentiable.

Thm (Liouville): If $f: \mathbb{C} \to \mathbb{C}$ is complex differentiable and bounded, then it is constant!

Applications 1.2: Complex analysis is used for:

1.2a) Powerful computational techniques

Evaluating sums

Thm: $\sum_{n=1} \frac{1}{n^2} = \frac{\pi^2}{6}$

Evaluate integrals

$$\int_0^\infty \frac{\log x}{(1+x^2)^2} dx = ? \qquad \int_0^{2\pi} \frac{\cos(t)^4 \cos(bt)}{\tan(5t)^4 (\sin 3t + \cos t)} dt$$

$$\int_0^\infty \frac{\sin ax}{\cos bx} \frac{x}{1+x^2} dx \qquad a, b \in \mathbb{R}.$$

Can all be evaluated with contour integration (part II).

1.2b) Study of Special functions: $\Gamma(z)$, Riemann $\zeta$-function, $\Theta$-functions, elliptic functions/integrals.

1.2c) Source of powerful computational techniques and examples in almost every area of math. Including

i) Number theory and automorphic forms
ii) Algebraic / Complex geometry
iii) Dynamics
iv) Real analysis + PDE (harmonic function, elliptic equations)
v) low dimensional geometry/topology (hyperbolic geometry,)
vi) geometric analysis (Dirac equations, cij geometry)
vii) Various applied math/algorithmic problems

## 1. ii) Basics of the Complex Numbers

Def 1.3 : The Complex plane is the vector space $(\mathbb{R}^2, i) = \mathbb{C}$. where $i \in \operatorname{End}(\mathbb{R}^2)$ is a counterclockwise rotation by $90°$, given by

$$i = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

Complex numbers may be written $z = x + iy$ where $x, y \in \mathbb{R}$.

Lemma 1.4 : Complex numbers obey normal algebraic properties with addition and multiplication using the $i^2 = -1$.

e.g. $$(x+iy)(c+id) = (cx - dy) + i(cy + dx).$$

Def 1.4 : The real and imaginary parts are

$$\operatorname{Re}(z) = x \quad \text{and} \quad \operatorname{Im}(z) = y \quad \text{when } x + iy = z.$$

$\bar{z} = x - iy$ is the complex conjugate

$$\operatorname{Re}(z) = \frac{z + \bar{z}}{2} \qquad \operatorname{Im}(z) = \frac{z - \bar{z}}{2i}$$

**Def 1.5**: The norm of a complex number is

$$|z| = \sqrt{Re(z)^2 + Im(z)^2} = \sqrt{x^2+y^2} = \sqrt{z\bar{z}}$$

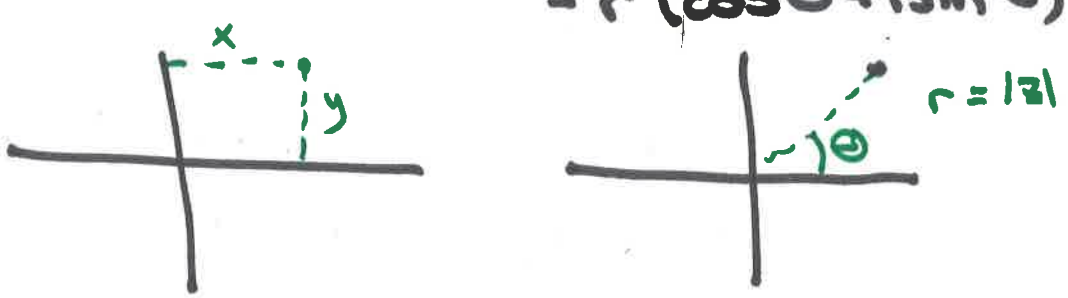
Note $|z|$ is the normal Euclidean distance on $\mathbb{R}^2$.

**Corollary 1.6**: The norm endows $\mathbb{C}$ with the standard metric space structure of $\mathbb{R}^2$. Thus the following notions make sense

- distance $|z-w|$
- open and closed sets
- Compact subsets
- Convergence $z_n \to z$.
- continuous, differentiable, and smooth functions $f: \mathbb{C} \to \mathbb{C}$.

**Def 1.7**: Points of $\mathbb{C}$ may also be written in polar form

$$z = x+iy = re^{i\theta}$$
$$= r(\cos\theta + i\sin\theta)$$



# Lecture 2 | Holomorphic Functions

Goal: Give several equivalent, increasingly non-obvious characterizations of what it means to be complex-differentiable.

## 2i: Complex Differentiability

$\Omega \subseteq \mathbb{C}$ open subset w/ (piecewise) smooth boundary

$f : \Omega \to \mathbb{C}$ a complex-valued function

**Def. 1**: $f : \Omega \to \mathbb{C}$ is said to be complex-differentiable or holomorphic if

$$\lim_{h \to 0} \frac{f(z+h) - f(z)}{h} = f'(z)$$

exists $\forall z \in \Omega$ for $h \in \mathbb{C}$. In this case we write

**Ex 2**: $f(z) = z$ is holomorphic

$f(z) = \sum_{n=1}^{N} a_n z^n$ is holomorphic.

**Rem 3**: holomorphicity is a very strong condition, since $h \to 0$ can be taken as the limit from any direction.

**Ex 4**: $f(z) = \bar{z} = x - iy$ is NOT holomorphic

$$f'(z) \stackrel{?}{=} \lim_{h \to 0} \frac{f(z+h) - f(z)}{h} = \frac{\bar{h}}{h} \quad \text{at } z = 0,$$

$$\frac{\bar{h}}{h} = \frac{x - iy}{x + iy} \quad \text{so} \lim_{h = (a,0)} = 1 \quad \text{but} \lim_{h = (0,b)} = -1.$$

**Lemma 5**: Suppose $f, g : \Omega \to \mathbb{C}$ are holomorphic. Then

a) $f + g$ is holomorphic and $(f+g)'(z) = f'(z) + g'(z)$

b) $fg$ is holomorphic and $(fg)'(z) = f'(z)g(z) + f(z)g'(z)$

c) If $f : \Omega \to U$ and $g : U \to \mathbb{C}$ then $g \circ f : \Omega \to \mathbb{C}$ is holomorphic and $(g \circ f)'(z) = g'(f(z)) f'(z)$.

**Proof**: Same as real case.

## 2.iii: The Cauchy-Riemann Equations

Given $f:\Omega \to \mathbb{C}$ (not necessarily holomorphic) denote by

$$\begin{array}{ccccc} f_{\mathbb{R}} : & \Omega & \longrightarrow & \mathbb{C} \\ & \cap & & \| \\ & \mathbb{R}^2 & & \mathbb{R}^2 \end{array}$$

the underlying real function

$$f_{\mathbb{R}}(x,y) = (\operatorname{Re} f(x+iy), \operatorname{Im} f(x+iy))$$
$$= (u(x,y), v(x,y)).$$

**Fact 6**: Let $A = \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ be a linear map. Then

$A\begin{pmatrix} x \\ y \end{pmatrix}$ is multiplication by a complex number $w = a+ib \in \mathbb{C}$

iff $A = \begin{pmatrix} a & -b \\ b & a \end{pmatrix}$. $\quad *$

**Proof**: $(a+ib)(x+iy) = (ax-by) + i(bx+ay)$.

**Thm 7**: A function $f:\Omega \to \mathbb{C}$ is holomorphic if and only if $f_{\mathbb{R}}:\Omega \to \mathbb{R}^2$ is differentiable and

$$\begin{cases} \dfrac{\partial u}{\partial x} = \dfrac{\partial v}{\partial y} \\[2ex] \dfrac{\partial u}{\partial y} = -\dfrac{\partial v}{\partial x} \end{cases}$$

holds.

**Proof**: The Jacobian is

$$df_{\mathbb{R}} = \begin{pmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{pmatrix}$$

$\Rightarrow$ If $f$ is holomorphic, $f'(z) \in \mathbb{C}$, so it has the form $*$.

$\Leftarrow$ Write $h = (h_1, h_2)$.

Then

$$u(x+h_1, y+h_2) = \frac{\partial u}{\partial x} h_1 + \frac{\partial u}{\partial y} h_2 + \mathcal{O}(|h|) + u(x,y)$$

$$v(x+h_1, y+h_2) = \frac{\partial v}{\partial x} h_1 + \frac{\partial v}{\partial y} h_2 + \mathcal{O}(|h|) + v(x,y)$$

so

$$\begin{aligned} f(z+h) - f(z) &= \left(\frac{\partial u}{\partial x} h_1 + \frac{\partial u}{\partial y} h_2\right) + i\left(\frac{\partial v}{\partial x} h_1 + \frac{\partial v}{\partial y} h_2\right) + \mathcal{O}(|h|). \\ &= \left(\frac{\partial u}{\partial x} h_1 + \frac{\partial u}{\partial y} h_2\right) + i\left(-\frac{\partial u}{\partial y} h_1 + \frac{\partial u}{\partial x} h_2\right) + \mathcal{O}(|h|). \\ &= \left(\frac{\partial u}{\partial x} - i\frac{\partial u}{\partial y}\right)\underbrace{(h_1 + ih_2)}_{=h} + \mathcal{O}(|h|). \end{aligned}$$

and

$$\lim_{h \to 0} \frac{f(z+h) - f(z)}{h} = \left(\frac{\partial u}{\partial x} - i\frac{\partial u}{\partial y}\right)(z). \quad ***$$

$\square$.

**Def 8**: The equations

$$\begin{cases} \dfrac{\partial u}{\partial x} = \dfrac{\partial v}{\partial y} \\ \dfrac{\partial u}{\partial y} = -\dfrac{\partial v}{\partial x} \end{cases} \quad \text{or} \quad \left(\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y}\right) + i\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right) = 0. \quad **$$

are called the Cauchy-Riemann Equations.

## 2.iii) Differential operators

Recall $z = x+iy$, $\bar{z} = x - iy$

**Def 9**: Define the $\partial, \bar{\partial}$ operators by

$$\partial = \frac{\partial}{\partial z} = \frac{1}{2}\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right)$$

$$\bar{\partial} = \frac{\partial}{\partial \bar{z}} = \frac{1}{2}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right)$$

⚠ the sign is opposite what occurs in $z, \bar{z}$!

<u>Lemma 10</u>: The Cauchy-Riemann equations are equivalent to $\bar{\partial} f = 0$. (thus being holomorphic!)

Moreover, when this holds the derivative is

$$f'(z) = \partial f.$$

<u>Proof</u>:

$$\begin{aligned}
\bar{\partial} f &= \bar{\partial}(u+iv) \\
&= \tfrac{1}{2}(\partial_x + i\partial_y)(u+iv) \\
&= \tfrac{1}{2}\left(\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y}\right) + \tfrac{1}{2} i \left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right) \\
&= **.
\end{aligned}$$

And if this holds

$$\begin{aligned}
\partial f &= \tfrac{1}{2}\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) + \tfrac{1}{2} i \left(\frac{\partial u}{\partial y} - \frac{\partial v}{\partial x}\right) \\
&= \tfrac{1}{2}\left(2\frac{\partial u}{\partial x}\right) + \tfrac{1}{2} i \left(2\frac{\partial u}{\partial y}\right) \\
&= \frac{\partial u}{\partial x} + i\frac{\partial u}{\partial y} = ***.
\end{aligned}$$

$\square$

<u>Def 11</u>
Recall

$$\Delta u = -\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)$$ is the <u>Laplacian</u>

and functions satisfying

$$\Delta u = 0$$ are called <u>Harmonic</u>

<u>Prop 12</u>: If $f$ is holomorphic $u, v$ are both harmonic.

$$\left\{ \begin{matrix} \text{holomorphic } f \\ \text{on } \Omega \end{matrix} \right\} \subseteq \left\{ \begin{matrix} \text{harmonic } f \\ \text{on } \Omega \end{matrix} \right\}$$

<u>Proof</u>: If $\bar{\partial} f = 0$ then

$$\begin{aligned}
0 = \partial\bar{\partial} f &= \tfrac{1}{2}(\partial_x - i\partial_y)(\partial_x + i\partial_y) f \\
&= \tfrac{1}{4}(\partial_x^2 + \partial_y^2 f) \\
&= -\tfrac{1}{4}\Delta f. = -\tfrac{1}{4}\Delta u - \tfrac{1}{4}\Delta v.
\end{aligned}$$

$\square$

<u>Rem 13</u>: $\Delta u = 0$ is a strong condition, the solution of a PDE. Many properties of holomorphic functions are special cases of general properties for (elliptic) PDEs.

# Lecture 3 Power Series Expansions.

Recall $f$ is holomorphic iff $\bar{\partial} f = 0$. In particular, $f(z) = z$

**3.i) Power Series**

$$\tfrac{1}{2}(\partial_x + i\partial_y)(x+iy) = \tfrac{1}{2}(1-1) = 0$$

is. By Lemma 2.5, if

$$f(z) = a_0 + a_1 z + \quad \ldots \quad + a_N z^N$$

then

$$\bar{\partial} f = 0$$

$$f'(z) = \partial f = a_1 + 2a_2 z + 3a_3 z^2 + \ldots N a_N z^{N-1}.$$

**Def 3.1**: Given a sequence $\{a_n\}_{n \in \mathbb{N}_0 \cup \{0\}}$, the corresponding power series centered at $z_0$ is

$$P_{z_0}(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n.$$

If there exists $R > 0$ st $P_{z_0}$ converges for $|z - z_0| < R$, the power series is convergent at $z_0$ with radius of convergence $R$.

**Def 3.2**: A function $f : \Omega \to \mathbb{C}$ is said to be analytic on $\Omega$ if $\forall z_0 \in \Omega$, there is a convergent power series such that

$$f(z) = P_{z_0}(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n$$

on $B_R(z_0)$.

**3.ii: Examples**: Many well known functions extend to (analytic) (holomorphic) functions via their Taylor series.

**Ex 3.3**:

$$f(z) = e^z = \sum_{n=0}^{\infty} \frac{z^n}{n!}$$

is analytic on $\Omega = \mathbb{C}$ by ratio test.

**Ex 3.4**: $f(z) = \sum_{n=0}^{\infty} z^n$

is analytic on $\{R<1\} = \Omega$ since

$$\sum_{n=0}^{\infty} |z|^n < \infty \quad \text{for } |z|<1$$
$$= \frac{1}{1-|z|} \quad \text{by geometric series.}$$

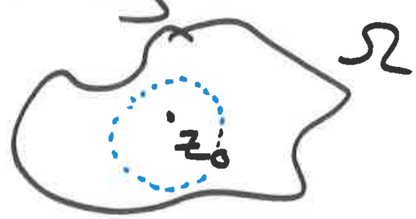
**Ex 3.5**:
$$\cos(z) := \sum_{n=0}^{\infty} (-1)^n \frac{z^{2n}}{(2n)!} = \frac{e^{iz}+e^{-iz}}{2}$$
$$\sin(z) := \sum_{n=0}^{\infty} (-1)^n \frac{z^{2n+1}}{(2n+1)!} = \frac{e^{iz}-e^{-iz}}{2i}$$

"Euler formulas"

**Prop 3.6**: Given a power series $P_{z_0}$, $\exists R \in [0,\infty]$ such that

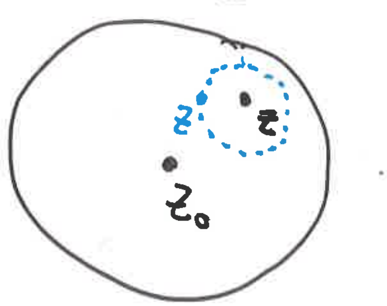
- $P_{z_0}$ converges absolutely for $|z-z_0|<R$
- $P_{z_0}$ diverges for $|z-z_0|>R$.

i.e. the domain of convergence must be




**Proof**: Define $\Lambda$ by $\Lambda = \limsup |a_n|^{\frac{1}{n}}$. Suppose $\Lambda \neq 0,\infty$ to begin.
~~second and first bullets are vacuous.~~

Set $R = \frac{1}{\Lambda}$ Then for any (fixed) $z$ w/ $|z-z_0|<R$, take $\delta>0$ small so




$z_1 \in B(z_0, (1-\delta)R)$.

then $$|z-z_0|^n < (1-\delta)^n R^n$$
$$|a_n| < \Lambda^n = \frac{1}{R^n}$$
so
$$|a_n(z-z_0)^n| < \frac{(1-\delta)^n R^n}{R^n} = (1-\delta)^n.$$
Comparison w/ geometric series shows convergence on $|z-z_0|<R$.
If $R > \frac{1}{\Lambda}$ then can extract a diverging subsequence $\square$.

**Thm 3.7**: A power series $P_{z_0}(z)$ defines a holomorphic function
$$f = P_{z_0}(z) : \Omega \to \mathbb{C}$$
for $\Omega = B(z_0, R)$. Moreover,
$$P_{z_0}'(z) = f'(z) = \sum_{n=0}^{\infty} n a_n z^{n-1} \quad \text{(and radius is same).}$$

**Corollary 3.8**: $f = P_{z_0}(z)$ is smooth on $\Omega$ and
$$\partial^{(k)} f = \sum_{n=0}^{\infty} n \cdot (n-1) \cdot \ldots \cdot (n-k+1)\, a_n z^{n-k}$$
$\square$.

**Proof**: Since $\lim_{n\to\infty} n^{1/n} = 1$
$$\limsup |a_n|^{1/n} = \limsup |n a_n|^{1/n}$$
so the radius of convergence is the same. It suffices to prove
$$\lim_{h\to 0} \frac{P_{z_0}(z+h) - P_{z_0}(z)}{h} = P_{z_0}'(z).$$
Since this implies holomorphicity.

Suppose $|z_1 - z_0| < r < R$. Write
$$P_{z_0}(z) = \underbrace{\sum_{n=0}^{N} a_n (z-z_0)^n}_{S_N} + \underbrace{\sum_{n=N+1}^{\infty} a_n (z-z_0)^n}_{E_N}$$

We claim that for $\varepsilon > 0$, $\exists\, \delta$ st for $|h| < \delta$,

$$\left| \frac{f(z_1+h) - f(z_1)}{h} - P_{z_0}'(z_1) \right| < \varepsilon .$$

Indeed, $\forall N \geq 1$.

$$\left| \frac{f(z_1+h) - f(z_1)}{h} - P_{z_0}'(z_1) \right| \leq \left| \frac{S_N(z_1+h) - S_N(z_1)}{h} - S_N'(z_1) \right| \quad *$$

$$+ \left| S_N'(z_1) - P_{z_0}'(z_1) \right| \quad **$$

$$+ \left| \frac{E_N(z_1+h) - E_N(z_1)}{h} \right| \quad *** .$$

- Since $S_N$ is finite, differentiating is okay. Choose $\delta$ st $* < \frac{\varepsilon}{3}$.
- Since $\lim_{N \to \infty} S_N'(z_1) = P_{z_0}'(z_1)$ since we are in R.o.C., take $N$ large so $** < \frac{\varepsilon}{3}$.
- $*** \leq \sum_{n=N+1}^{\infty} |a_n| \left| \frac{(z_1+h)^n - z_1^n}{h} \right| \leq \sum_{n=N+1}^{\infty} |a_n| \cdot n \cdot r^{n-1}$

$$\leq \frac{\varepsilon}{3} \text{ for } N \text{ large}$$

(by convergence of $P'$)

since $(z_1+h)^n - z_1^n = h(h^{n-1} + h^{n-2}z_1 + \dots z_1^{n-1})$
$\leq h(n \cdot |z_1|^{n-1})$. $\square$

<u>Remark 3.9</u>: For a general $f: \Omega \to \mathbb{R}^2$ real, $z = x+iy$, $\bar{z} = x - iy$
give a Taylor series $f(x,y) = \sum_{n=0}^{\infty} \sum_{k=0}^{n} a_{nk} z^k \bar{z}^{n-k}$.

holomorphicity means there are no $\bar{z}$ terms, so $a_{nk} = 0$ except $k = n$.

This is now clearly a very restrictive condition.

# Lecture 4 | Integration I: domains + primitives.

Rem 4.0: We are going out of order 1.3 $\rightsquigarrow$ 3.5 $\rightsquigarrow$ 2.1–2.2.

## 4i) Homotopy and Simply Connected Domains

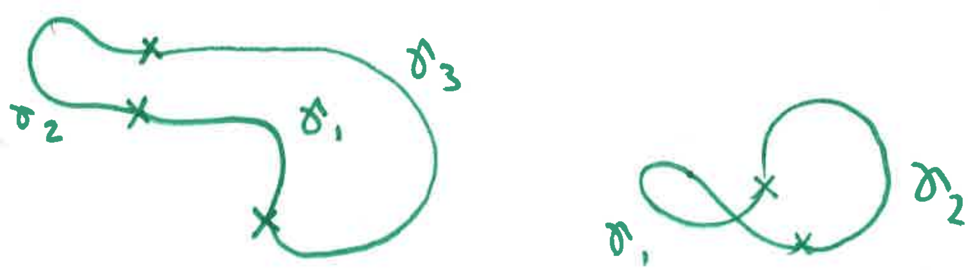
**Def 4.1**: A (piecewise smooth) parameterized curves is a function $\gamma : [a,b] \longrightarrow \Omega$ such that $\exists N < \infty$ and intervals

$$[a,b] = [\underset{\overset{\|}{a}}{a_1}, a_2] \cup [a_2, a_3] \ldots \cup [a_{n-1}, \underset{\overset{\|}{b}}{a_n}].$$

such that $\gamma|_{[a_{i-1},a_i]} : [a_{i-1}, a_i] \longrightarrow \Omega$

is $C^\infty$, and $\gamma$ is continuous.

**Ex 4.2**




**Def 4.3**: Two parameterized curves $\gamma_1, \gamma_2$ are said to be equivalent (or isomorphic) if $\exists$ a smooth diffeomorphism (bijection, smooth piecewise)

$\varphi : [a_1, b_1] \xrightarrow{\sim} [a_2, b_2]$ such that

$$\gamma_2(\varphi(t)) = \gamma_1(t).$$

**Def 4.4**: A (piecewise smooth) curve is an equivalence class

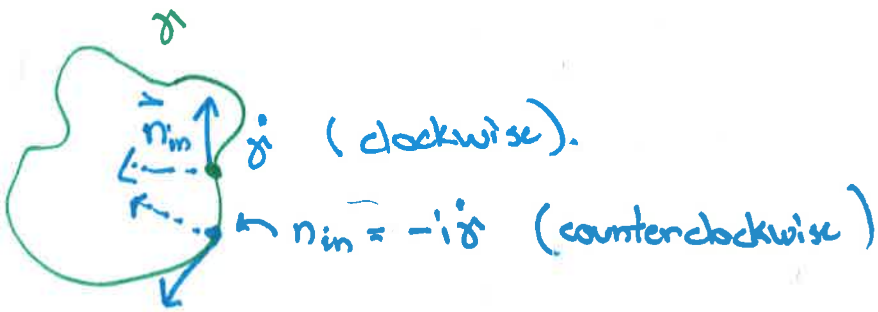
$$[\gamma] \in \{\text{Parameterized curves}\}/\simeq .$$

**Def 4.5**: • An orientation is a choice of direction of $\dot{\gamma}$ along $\gamma$.

• A curve is closed if $\gamma(a) = \gamma(b)$.

• A closed curve is clockwise oriented if

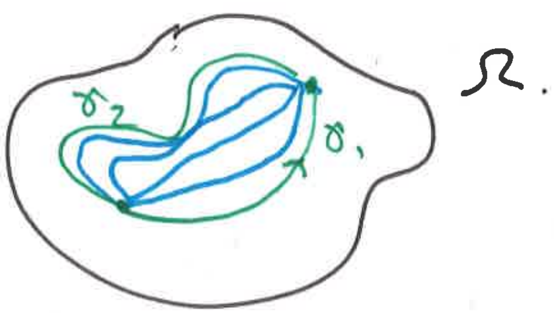
$$\vec{n}_{\text{inward}} = i\dot{\gamma}$$




**Def 4.6**: Two curves are homotopic if $\exists$ a continuous function

$$\Gamma : [0,1] \times [a,b] \longrightarrow \Omega$$

with
$$\Gamma\big|_{\{0\}\times[a,b]} = \gamma_1$$
$$\Gamma\big|_{\{1\}\times[a,b]} = \gamma_2.$$

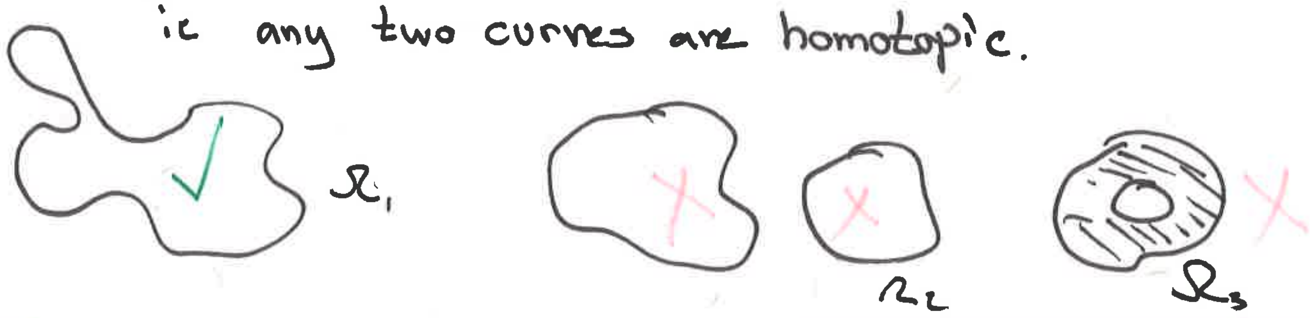



**Def 4.7**: A domain $\Omega \subseteq \mathbb{C}$ is simply connected

if $\pi_1(\Omega) = \{\text{curves in } \Omega\} \big/ \gamma_1 \underset{\text{hmty}}{\sim} \gamma_2$

$= \text{pt}.$

ie any two curves are homotopic.



# 4.ii) Line Integrals



Recall a line integral over $\gamma = (\gamma_x(t), \gamma_y(t))$

$$\int_\gamma M\,dx + N\,dy = \int_a^b \left[ M(\gamma)\cdot \frac{d\gamma_x}{dt} + N(\gamma)\frac{d\gamma_y}{dt}\right] dt$$

**Def 4.10**: the (complex line) integral of $f$ over $\gamma$ (not necessarily holomorphic) if

$$= \int_\gamma f(z)\,dz = \int_\gamma (u+iv)(dx+idy)$$

$$= \int_\gamma u\,dx - v\,dy + i\int_\gamma v\,dx + u\,dy$$

$$= \int f(\gamma(t))\,\gamma'(t)\,dt$$

**Prop/Def 4.11**: $f$ has a primitive over $\Omega$ if $\exists F$ st $F'(z) = f$. In this case

$$\int_\gamma f(z)\,dz = F(\gamma(b)) - F(\gamma(a))$$

by FToC.

$= 0$ if $\gamma$ is closed. $\square$.

**Ex 4.12**: $f(z) = \frac{1}{z}$ does not have a primitive, since $\gamma = e^{it}$ for $t\in[0,2\pi]$ has

$$\int_\gamma \frac{1}{e^{it}}\cdot ie^{it}\,dt = 2\pi i \neq 0.$$

**Thm 4.13** (Cauchy's Thm): If $\Omega \subseteq \mathbb{C}$ is simply connected, and $f : \Omega \to \mathbb{C}$ is holomorphic, then

$$\oint_{\gamma} f(z)\,dz = 0$$

for any closed curve $\gamma \subseteq \Omega$.

Proof has two steps.

1) If $\Omega$ is simply connected, $\exists\, \varphi : \underset{B(0,1) \subseteq \mathbb{R}^2}{D} \to \Omega$ such that $\varphi|_{\partial D} = \gamma$.

**Proof**: Let $\gamma_0 = \gamma$ and $\gamma_1 = z_0$ constant. By simply connected, $\exists\, \Gamma : [0,1] \times [0, 2\pi]$ st $\Gamma(0,t) = \gamma_0$, $\Gamma(1,t) = z_0$.


$$\varphi(r, \theta) = \begin{cases} z_0 & |r| \leq \frac{1}{2} \\ \Gamma(2r-1, \theta) & |r| \geq \frac{1}{2}. \end{cases}$$

□.

2) Recall Green's theorem. If $\gamma = \partial D$ then

$$\oint_{\gamma} M\,dx + N\,dy = \iint_D \left( \frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right) dx\,dy.$$

**Proof**: Let $D$ be as in step 1.

$$\int_{\gamma} f(z)\,dz = \int_{\gamma} u\,dx - v\,dy + i \int_{\gamma} v\,dx + u\,dy$$

$$= \int_D \underbrace{\left( -\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} \right)}_{=0} dx\,dy + \int_D \underbrace{\left( \frac{\partial u}{\partial x} - \frac{\partial v}{\partial y} \right)}_{=0} dx\,dy$$

by Cauchy-Riemann

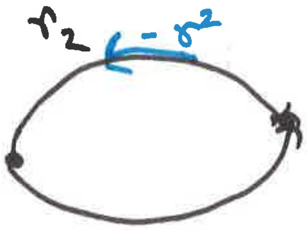
□.

**Corollary 4.14:** If $\Omega$ is simply connected and $\gamma_1$, $\gamma_2$ have same endpoints, then

$$\oint_{\gamma_1} f\,dz = \oint_{\gamma_2} f(z)\,dz$$

for $f$ holomorphic.

**Proof:**




$$\oint_{\gamma_2} f(z)\,dz = -\int_{-\gamma_2} f(z)\,dz$$

opposite orientation

Set $\gamma = \gamma_1 - \gamma_2$

$\Rightarrow$

$$0 = \oint_{\gamma} f(z)\,dz = \int_{\gamma_1} f\,dz - \int_{\gamma_2} f\,dz$$

$\square$.

**Corollary 4.15:** If $f : \Omega \to \mathbb{C}$ is holomorphic, and $\Omega$ simply connected, $\exists$ a primitive

$$F : \Omega \to \mathbb{C}$$

for $f$. Any two differ by a constant.

**Proof:** Fix $z_0 \in \Omega$. Set $F(z) = \int_{\gamma} f(z)\,dz$

where $\gamma(0) = z_0$, $\gamma(1) = z$. Independent of path by previous corollary. By FToC $F'(z) = f(z)$.

If $F_1' = F_2'$ then differ by a complex constant $\square$.

# $\boxed{\text{Lecture 5}}$ Cauchy's Integral Formula I

Recall if $f : \Omega \to \mathbb{C}$ is holomorphic then

$$\oint_\gamma f(z)\,dz = 0. \qquad *$$

For any closed curve $\gamma \subset \Omega$. If $f(z)$ is not holomorphic this is <u>false</u>!

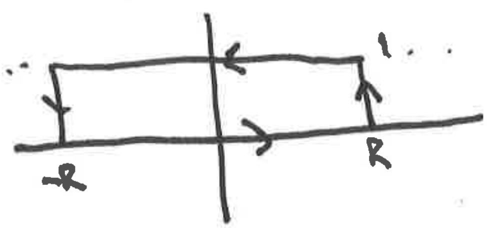
Point singularity at center $\Rightarrow$ Cauchy's Integral formula

$\Rightarrow$ Equivalent condition of holomorphicity.

<u>Ex 5.1</u>: If $\gamma = \gamma_1 \cup \gamma_2$ then $(*)$ says that

$$0 = \int_{\gamma_1} f(z)\,dz + \int_{\gamma_2} f(z)\,dz.$$

5.1a) : $\Omega =$

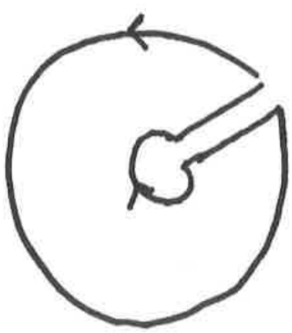



$$\int_{\text{bottom}} f(z) + \int_{\text{right}} f(z)\,dz$$

$$0 = -\int_{\text{top}} f(z)\,dz - \int_{\text{left}} f(z)\,dz.$$

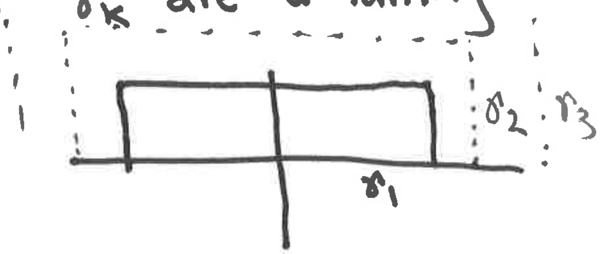
5.1b) :

$\Omega =$



$$0 = -\int_{\text{inner}} f(z)\,dz + \int_{\text{outer}} f(z)\,dz + \int_{\text{radial}} f(z)\,dz.$$

5.1c) If. $\gamma_k$ are a family of curves




$$0 = \int_{b_k} f(z)\,dz + \dots + \int_{\ell_k} f(z)\,dz$$

If $\lim_{k\to\infty} \int_{\text{segment}} f(z)\,dz = 0$, then integral may be evaluated by other segments.

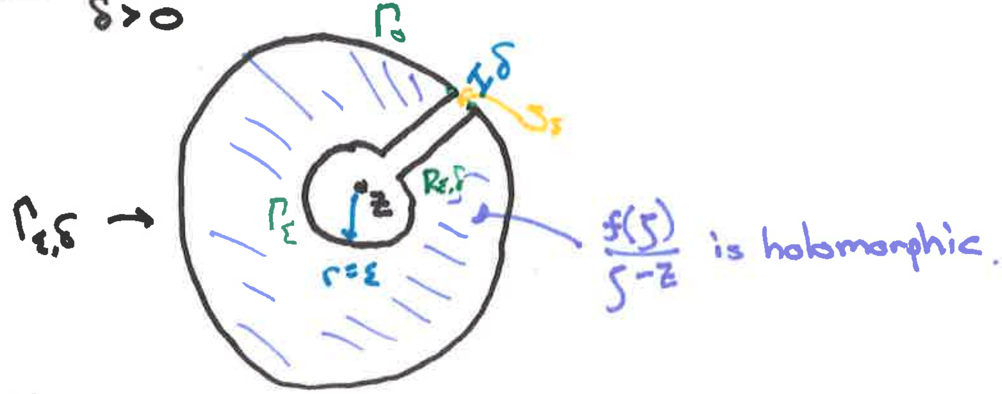
## 5.ii: Proof of the Integral Formula

**Theorem 5.2** (Cauchy Integral Formula)
Suppose $D \subseteq \Omega$ is a disk with $\partial D = \gamma$. If $f : \Omega \to \mathbb{C}$ is holomorphic

$$f(z) = \frac{1}{2\pi i} \int_\gamma \frac{f(\zeta)}{\zeta - z}\, d\zeta$$

for any $z \in D$.

**Proof**: Let $\varepsilon > 0$, $\delta > 0$ be small, and consider the curve




by Thm 4.13

$$0 = \int_{\Gamma_{\varepsilon,\delta}} \frac{f(\zeta)}{\zeta - z}\, d\zeta$$
$$= \underbrace{\int_{\Gamma_0 \setminus S_\delta} \frac{f(\zeta)}{\zeta - z}\, d\zeta}_{1} - \underbrace{\int_{\Gamma_\varepsilon} \frac{f(\zeta)}{\zeta - z}\, d\zeta}_{2} + \underbrace{\int_{R_{\varepsilon,\delta}} \frac{f(\zeta)}{\zeta - z}\, d\zeta}_{3}$$

Claims:

$$\lim_{\varepsilon,\delta \to 0} 1 = \int_\gamma \frac{f(\zeta)}{\zeta - z}\, d\zeta$$

$$\lim_{\varepsilon,\delta \to 0} 2 = 2\pi i\, f(z)$$

$$\lim_{\varepsilon,\delta \to 0} 3 = 0$$

$\Rightarrow 2\pi i f(z) = \int_\gamma \frac{f(\zeta)}{\zeta - z}\, dz$ so done.

Since $f$ is holomorphic (so continuous)

$$\int_{S_\delta} \underbrace{\frac{f(\zeta)}{\zeta - z}}_{F_z(\zeta)} d\zeta \leq C\delta \operatorname{Sup}\left|\frac{f(\zeta)}{\zeta - z}\right| \to 0. \quad ① \checkmark$$

$$\int_{R_{\varepsilon,\delta}} F_z(\zeta) d\zeta = \int_{R_1} F_z(\zeta) d\zeta - \int F_z(\zeta + O(\delta)) d\zeta$$

$$= 0 + O(\delta). \quad ③ \checkmark$$

By same argument as ①

$$\int_{\Gamma_\varepsilon} \frac{f(\zeta)}{\zeta - z} d\zeta = \int_{r=\varepsilon} \frac{f(\zeta)}{\zeta - z} d\zeta = f(z) \int_0^{2\pi} \frac{1}{\varepsilon e^{i\theta}} d(z + \varepsilon e^{i\theta}) + \underbrace{\int_0^{2\pi} \frac{f(\zeta) - f(z)}{\zeta - z}}_{④}$$

$$= f(z) \int_0^{2\pi} \frac{i\varepsilon e^{i\theta}}{\varepsilon e^{i\theta}} d\theta + ④$$

$$= 2\pi i f(z) + ④$$

And $|f(\zeta) - f(z)| \leq C\varepsilon |\operatorname{Sup} f'|$ so

$$④ \leq C\varepsilon |\operatorname{Sup} f'| \underbrace{\int_0 \frac{1}{\zeta - z} d\zeta}_{< \infty.} \xrightarrow{\varepsilon \to 0.} 0$$

$\square$

## 5.iii) Corollaries of the Integral Formula

**Corollary 5.3** : With $D, \gamma, \Omega, f : \Omega \to \mathbb{C}$ as above

$$\left(\frac{\partial}{\partial z}\right)^n f(z) = \frac{n!}{2\pi i} \int_\gamma \frac{f(\zeta)}{(\zeta - z)^{n+1}} d\zeta$$

for $z \in D$.

**Proof** : Thm 5.2 is $n = 0$. For $n \geq 1$

$$\frac{\partial}{\partial z} \int_\gamma \frac{f(\zeta)}{(\zeta - z)^n} d\zeta = \int_\gamma \frac{\partial}{\partial z} \frac{f(\zeta)}{(\zeta - z)^n} d\zeta$$

since integrand is $C^1$ on $\gamma$.

Corollary 5.4 : Cauchy's Integral formula is if and only if.

Proof : $f$ holomorphic $\Rightarrow$ Integral formula by Thm 5.2

Assume
$$f(z) = \frac{1}{2\pi i}\int_\gamma \frac{f(\zeta)}{\zeta - z}\,d\zeta$$

then
$$\frac{\partial}{\partial \bar{z}} f(z) = \frac{1}{2\pi i}\int_\gamma \frac{\partial}{\partial \bar{z}} \frac{f(\zeta)}{(\zeta - z)}\,d\zeta = 0$$

because $\frac{1}{\zeta - z}$ is holomorphic (on $\gamma$). □

Recall $f(z) = \sum a_n (z - z_0)^n \Rightarrow f$ is holomorphic.

Thm 5.5 : Suppose $f : \Omega \to \mathbb{C}$ is holomorphic. Then if $D \subseteq \Omega$ is a disk, $f$ has a convergent power series
$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n \qquad a_n = \frac{f^{(n)}(z_0)}{n!}$$
on $D$ for $z_0$ the center.

Proof : Write
$$\frac{1}{\zeta - z} = \frac{1}{(\zeta - z_0) - (z - z_0)}$$
$$= \frac{1}{\zeta - z_0} \cdot \frac{1}{1 - \left(\frac{z - z_0}{\zeta - z_0}\right)}.$$

Since $z \in D$, and $\zeta \in \partial D$, $\frac{|z - z_0|}{|\zeta - z_0|} < 1$, so
$$\frac{1}{1 - \left(\frac{z - z_0}{\zeta - z_0}\right)} = \sum_{n=0}^{\infty} \frac{(z - z_0)^n}{(\zeta - z_0)^n}$$

$$f(z) = \frac{1}{2\pi i}\int_\gamma \frac{f(\zeta)}{\zeta - z}\,d\zeta = \frac{1}{2\pi i}\int_\gamma \frac{f(\zeta)}{(\zeta - z_0)} \cdot \frac{1}{(\zeta - z_0)^n}\,d\zeta \cdot (z - z_0)^n$$
$$= \frac{f^{(n)}(z_0)}{n!} \cdot (z - z_0)^n$$

□

Corollary 5.6: The ratio of successive Fibonacci numbers is
$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \gamma \qquad \left(\gamma = \frac{1+\sqrt{5}}{2} \text{ the "Golden ratio"}\right).$$

Proof: Define $f(z) = \sum_{n=0}^{\infty} a_n z^n$ an Fibonacci. Recurrance becomes
$$f(z) = (z+z^2)f(z)+1$$
$$\Rightarrow f(z) = \frac{1}{1-z-z^2}.$$
This has a singularity at $\frac{1}{\gamma}$. By proof of Thm 3.6
$$\limsup |a_n|^{1/n} = \frac{1}{1/\gamma} = \gamma. \qquad \square$$

Rem 5.7: (Digression on Piscot numbers)
$$|d(\gamma^n, \mathbb{Z})| \xrightarrow{n\to\infty} 0$$
exponentially fast, hence $\gamma$ is a Piscot number. It is an open question whether any $\alpha$ with this property is algebraic.

# Lecture 6: Properties of Holomorphic Functions

Let's summarize the different characterizations of holomorphic:

**Thm 6.1**: The following are all equivalent, for a function $f: \Omega \to \mathbb{C}$

1) $f$ is complex differentiable with continuous derivative.
2) $f$ satisfies $\bar{\partial} f = 0$ the Cauchy-Riemann equations.
3) $f$ is analytic with local, convergent power series
$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n$$
centered at $z_0$, with only holomorphic terms (ie no $\bar{z}$).
4) $f$ satisfies the Cauchy integral formula
$$f(z) = \frac{1}{2\pi i} \int_\gamma \frac{f(\zeta)}{\zeta - z} \, d\zeta$$
for $\gamma = \partial D$ with $D \subseteq \Omega$.

**Proof**: 1) $\Longleftrightarrow$ 2) Lecture 2
3) $\Longrightarrow$ 1) Lecture 3
2) $\Longleftrightarrow$ 4) Lecture 4-5
3) $\Longrightarrow$ 1) Lecture 5 (using Cauchy integral formula) $\square$

## 6.ii Consequences of holomorphicity

**Thm 6.2** (Elliptic Regularity)
Suppose $f: \Omega \to \mathbb{C}$ is continuously differentiable, and $\bar{\partial} f = 0$. Then $f$ is smooth (infinitely differentiable).

**Proof**: Immediate from characterizations 3) and 4) e.g.
$$f^{(n)}(z) = \frac{n!}{2\pi i} \int_\gamma \frac{f(\zeta)}{(\zeta - z)^{n+1}} \, d\zeta$$
$\square$

Rem 6.3 : Thm 6.2 is an instance of a very general property of "elliptic" PDEs, which $\bar{\partial}$ is an example of. The general proof is harder (Math 205a/b) but can be done with similar methods.

Thm 6.4 : If $f : \Omega \to \mathbb{C}$ for $\Omega$ connected is holomorphic, then the zeros of $f$ are isolated unless $f \equiv 0$!

Proof : If not, this would contradict

Lemma 6.5 : If $f$ is holomorphic on $\Omega$ (connected) and $f^{-1}(0)$ has an accumulation point. Then $f \equiv 0$.

Proof : Suppose $f(z_0) = 0$ and $f(w_k) = 0$ w/ $w_k \to z_0$. First Assume $\Omega = D$.
Write

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n.$$

If $f \not\equiv 0$, then $\exists$ a minimal $n_0$ so that $a_n \neq 0$,
(and $a_0 = 0$ iff $f(z_0) = 0$)

hence

$$f(z) = a_{n_0} (z - z_0)^{n_0} (1 + g(z - z_0))$$

where $g(z - z_0) = \sum_{m=1}^{\infty} b_m (z - z_0)^m$ vanishes at $z_0$. Since $a_{n_0}(z - z_0)^{n_0} \neq 0$ except at $z_0$, we would have to have

$$1 + g(w_k - z_0) = 0.$$

but $g(w_k - z_0) \to 0$ as $w_k \to z_0$, a contradiction.

ii) Suppose $\Omega$ is connected. Let $U \subseteq \Omega$ be the set so that $U = f^{-1}(0)$.
Then the above shows $U$ is open, a by continuity $U$ is closed.
Hence

$$U = \Omega.$$

$\square$

<u>Thm 6.6</u> (Analytic Continuation) Suppose $f, g : \Omega \to \mathbb{C}$ ($\Omega$ connected) are holomorphic and $\exists \{x_k\}$ with an accumulation point s.t.

$$f(x_k) = g(x_k)$$

$\forall k$. Then $f \equiv g$ on $\Omega$.

<u>Proof</u>: Apply Lemma 6.5 to $f - g$.

<u>Thm 6.7</u> (Liouville): If $f : \mathbb{C} \to \mathbb{C}$ is bounded and holomorphic, then it is constant.

<u>Proof</u>: By Cauchy's Integral Formula, $\forall R > 0$.

$$f(0) = \frac{1}{2\pi i} \int_{\partial D_R} \frac{f(\zeta)}{(\zeta - 0)^2} \, d\zeta$$

$$\leq \frac{\sup |f(\zeta)|}{2\pi} \int_{\partial D_R} \frac{1}{\zeta^2} \, d\zeta \leq \frac{C}{R}.$$

$\square$

<u>Rem 6.8</u>: Not all the above are properties of holomorphic functions:

$$\underset{\text{holomorphic}}{\mathcal{H}(\Omega : \mathbb{C})} \subset \left\{ \begin{matrix} \text{sol. of elliptic} \\ \text{equations} \end{matrix} \right\} \subset \underset{\text{analytic}}{\mathcal{C}^\omega(\Omega : \mathbb{C})}$$

- elliptic regularity
- integral formula
- maximum principle (next)

- analytic continuation

<u>Rem 6.9</u>: Everything holds equally for "anti-holomorphic" functions, and can be obtained by

$$\partial \bar{f} = \overline{\bar{\partial} f}$$

Choosing to focus on holomorphic is a simple (human) convention.

**Thm 6.10** : (The maximum principle) A non-constant holomorphic $f:\Omega \to \mathbb{C}$ attains its maximum (modulus) on $\partial\Omega$.

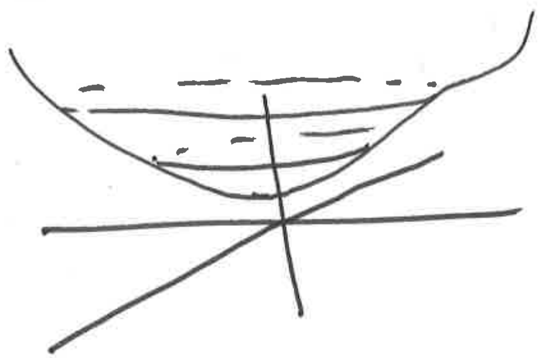
**Proof** : Suppose $z_0$ is an interior max (of $|f(z_0)|$), and take $B_R(z_0) \subseteq \Omega$ for some $R$.

$$f(z_0) = \frac{1}{2\pi i}\int_{\partial B_R} \frac{f(\zeta)}{\zeta - z_0}\,d\zeta$$

$$= \frac{1}{2\pi i}\int_{\partial B_R} \frac{f(Re^{i\theta})}{Re^{i\theta}}\, iRe^{i\theta}\,d\theta = \frac{1}{2\pi}\int_{\partial B_R} f(z_0 + Re^{i\theta})\,d\theta \le f(z_0)$$

with equality iff $f(z_0+Re^{i\theta}) = f(z_0)$ $\forall \theta \in S^1$. □

Re(f).



## 6.iii : Applications

**Thm 6.11** (Fundamental Thm of Calculus) Every polynomial $P(z) = a_1 + a_2 z + \dots + a_n z^n$ has a root in $\mathbb{C}$. (hence $n$ by factoring)

**Pf** : Since $z^n$ eventually dominates $\lim_{R\to\infty} \frac{1}{|P(Re^{i\theta})|} = 0$.

Hence if there are no roots, $\frac{1}{P(z)}$ is a bounded holomorphic function, contradiction. □

# Lecture 7 Contour Integration I: holomorphic case.

Next main goal: use properties of holomorphic function to develop computational techniques.

## 7i: Examples of Integration

**Ex 7.1**: Compute $\int_0^\infty \frac{1-\cos x}{x^2}\,dx$ ← note $1-\cos x \approx x^2$ at $x \to 0$ so this is finite.

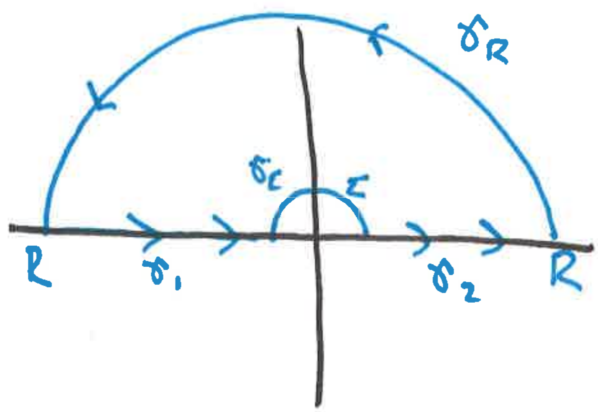
**Def 7.2**: a "contour" is another name for a piecewise differentiable closed curve.

**Step 1**: Extend $f(x) = \frac{1-\cos(x)}{x^2}$ to $\mathbb{C}$ by

$$f(z) = \frac{1-e^{iz}}{z^2}$$

so $f(x) = \operatorname{Re} f(z)$ along $\mathbb{R} \subseteq \mathbb{C}$.

**Step 2**: Choose a contour




**Step 3**: Apply Cauchy's Integral Formula

$$0 = \oint_\Gamma f(z)dz = \underbrace{\int_{-R}^{-\varepsilon} f(z)dz}_{1} + \underbrace{\int_\varepsilon^R f(z)dz}_{1'} + \underbrace{\int_{\gamma_\varepsilon} f(z)dz}_{2} + \underbrace{\int_{\gamma_R} f(z)dz}_{3}.$$

(3) $\left|\int_{\gamma_R} f(z)dz\right| \leq i\int_0^\pi \frac{|1-e^{iRe^{i\theta}}|}{|Re^{i\theta}|^2} \cdot Re^{i\theta}\,d\theta$

$= C\int_0^\pi \frac{2}{R}\,d\theta \leq \frac{C}{R} \to 0.$ $\quad e^{iR\cos\theta + i\sin\theta}$ $\quad e^{iR\cos\theta}e^{-\sin\theta}$

(2) $\left\{\int_{\gamma_\varepsilon} f(z)dz\right\} \lessapprox \; = i\int_\pi^0 \frac{-iz}{z^2}\,\varepsilon e^{i\theta}\,d\theta + O(\varepsilon)$

$= \int_\pi^0 d\theta = -\pi$

(1) $\int_0^\varepsilon f(x)dx \longrightarrow 0$ since $f(x)$ bounded

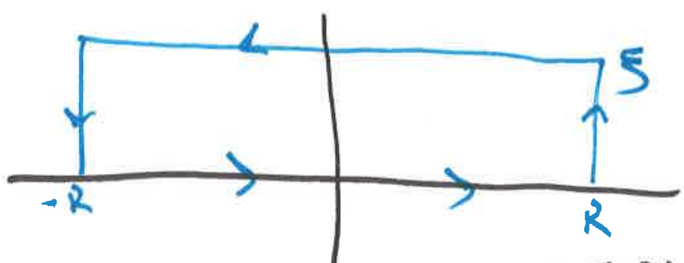
$\int_R^\infty f(x)dx \longrightarrow 0$ $\quad f = O(\frac{1}{x^2}).$

$0 = \lim_{R,\varepsilon} \int_\varepsilon^R f(x)dx + \int_{-R}^{-\varepsilon} f(x)dx + 0 - \pi$

$= 2\int_0^\infty f(x)dx - \pi$ $\qquad\qquad$ □.

<u>Ex 7.3</u> : Compute $\int_{-\infty}^\infty e^{-\pi x^2} e^{-2\pi i x\xi}\,dx$ for $\xi \in \mathbb{R}$.

Let $f(z) = e^{-\pi z^2}$, and consider




$0 = \lim_{R\to\infty} \int_{-R}^R e^{-\pi x^2}dx + \int_0^\xi e^{-\pi(R^2 + 2iRy - y^2)}\,i\,dy + \int_{+R}^{-R} e^{-\pi(x+i\xi)^2}dx$

$= \underset{\text{(Gaussian Integral)}}{1} \quad + \text{(same)} + \underbrace{\lim_{R\to 0} e^{-\pi R^2} \int_0^\xi \text{bounded}\,dy}_{=0} + \underline{\qquad\qquad}$

$\leftarrow \int_R^{\infty} f(z)\,dz \to 0.$

$$0 = 1 - \int_{-\infty}^{\infty} e^{\pi\xi^2} e^{-\pi x^2} e^{-2\pi i \xi x}\,dx$$

$$\Rightarrow \int_{-\infty}^{\infty} e^{-\pi x^2} e^{-2\pi i \xi x}\,dx = e^{-\pi\xi^2}. \qquad \square$$

<u>Rem 7.4</u> : We have shown that

$$\mathcal{F}(e^{-\pi x^2}) = \hat{f}(\xi) = e^{-\pi\xi^2}$$

is its own Fourier transform. This is a fundamental property of the Gaussian, and is related to Heisenberg's uncertainty principle in Quantum Mechanics.

## 7.ii) Intuition for Residues

Let $\Gamma = \{|z|=1\}$ be the unit circle, let $f(z) = z^n$ for $n \in \mathbb{N}$.

$$\int_\Gamma f(z)\,dz = i\int_0^{2\pi} e^{in\theta} e^{i\theta}\,d\theta$$

$$= i\int_0^{2\pi} \cos(n+1)\theta + i\sin(n+1)\theta\,d\theta$$

$$= \begin{cases} 0 & n \neq -1 \\ 2\pi i & n = -1. \end{cases}$$

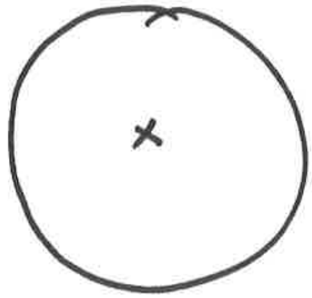
$\Rightarrow$ If $n \geq 0$ this follows from Cauchy's Integral formula, but $\frac{1}{z^m}$ is not holomorphic. Still, the integral formula holds for $m \neq 1$.

<u>Conclusions</u> i) Can evaluate integrals over ↺ contour by only looking at $\frac{1}{z}$ term in power series.

ii) Note $z^n = d\left(\frac{1}{n+1} z^{n+1}\right)$ has a primitive

so $\oint_\Gamma z^n\,dz = 0$ unless $n = -1$ ($\log z$). something weird w/ complex log!

Remark : This is ultimately a topological fact.



How many 1-forms are there on $D^2 \setminus \{0\}$ such that $d(f(z)dz) = \bar{\partial} f \, dz \wedge d\bar{z} = 0$ (holomorphic) but $f \neq dg$ for $g : D^2 \setminus \{0\} \to \mathbb{C}$.

$$\{\operatorname{Ker} d\} / \operatorname{Im} d = H^1_{dR}(D^2 \setminus \{0\}; \mathbb{C}) = \mathbb{C}$$

and the Hodge theorem says $\frac{1}{z} dz$ is the unique harmonic representative.

# Lecture 8: Meromorphic Functions and Poles



If $f : D \setminus \{0\} \to \mathbb{C}$ fails to be holomorphic at $0$, it can be

1) removable singularity (eg $f(z) = 1$ on $D \setminus \{0\}$)
2) pole (eg $f(z) = \frac{1}{z^n}$ )
3) essential singularity (eg $f(z) = e^{\frac{1}{z}}$ )

**Def 8.1**: a holomorphic function $f : \Omega \to \mathbb{C}$ is said to have an isolated singularity at $z_0$ if $\exists\, D \setminus \{0\} \subseteq \Omega$ w/ $0$ at $z_0$, and $f$ holomorphic on $D \setminus \{0\}$.

**Def 8.2**: If $f : \Omega \to \mathbb{C}$ is holomorphic, it has

- a zero at $z_0$ if $f(z_0) = 0$
- a pole at $z_0$ if $\frac{1}{f}$ is holomorphic near $z_0$, and $\frac{1}{f}$ has a zero.

**Lemma 8.3**: Suppose $f : \Omega \to \mathbb{C}$ is holomorphic and $f(z_0) = 0$. If $f \not\equiv 0$, then $\exists!\, n \in \mathbb{N}$ and $g : \Omega \to \mathbb{C}$ w/ $g$ holomorphic and $g(z_0) \neq 0$ such that

$$f(z) = (z - z_0)^n g(z).$$

**Proof**: Write

$$f(z) = \sum_{k=0}^{\infty} a_k (z - z_0)^k.$$

Take $n$ to be the smallest nonzero $a_k$. Then

$$f(z) = \sum_{k=n}^{\infty} a_k (z - z_0)^k = (z - z_0)^n \underbrace{\sum_{k=n}^{\infty} a_k (z - z_0)^{k-n}}_{:= g(z)}$$

, nonvanishing

$n$ is clearly unique as $(z-z_0)^n g(z) = (z-z_0)^m h(z)$ cannot have both $g, h$ non-vanishing. $\square$

**Lemma 8.4**: If $f : \Omega \setminus \{z_0\} \to \mathbb{C}$ is holomorphic with a pole at $z_0$, $\exists!\ n \in \mathbb{N}$, $h : \Omega \to \mathbb{C}$ holomorphic such that

$$f(z) = \frac{1}{(z-z_0)^n} h(z).$$

**Def 8.4'**: $n$ is called the order of the pole

**Proof**: $\frac{1}{f}$ has a zero of order $n$, so $\frac{1}{f} = (z-z_0)^n g(z)$. The result follows for $h = \frac{1}{g}$ $\square$.

**Def 8.5**: A function $h : \Omega \to \mathbb{C}$ is said to be meromorphic if $\exists \{z_1 \ldots\}$ w/o accumulation points s.t. $h : \Omega \setminus \{z_1, z_2, \ldots\} \to \mathbb{C}$ is holomorphic and $h$ has poles at each $z_i$.

Note: $h$ is only defined on $\Omega \setminus \{z_i\}$ but we still say its meromorphic on $\Omega$.

**Prop 8.6**: At a pole of order $n$, a meromorphic function has a convergent expansion

$$h(z) = \frac{a_{-n}}{(z-z_0)^n} + \frac{a_{-n+1}}{(z-z_0)^{n-1}} + \cdots + \frac{a_{-1}}{(z-z_0)} + G(z)$$

holomorphic

$$= \sum_{k=-n}^{\infty} a_k (z-z_0)^n$$

**Proof**: $h(z) = \frac{1}{(z-z_0)^n} g(z)$ by lemma 8.4,

$$= \frac{1}{(z-z_0)^n} \sum_{k=0}^{\infty} a_k (z-z_0)^k \qquad \square$$

**Def 8.7**: The negative terms

$$\frac{a_{-n}}{(z-z_0)^n} + \cdots + \frac{a_{-1}}{(z-z_0)}$$

are called the principal part.

**Def 8.8**: The coefficient of the order $-1$ term is called the <u>residue</u> of $h$ at $z_0$. Denoted L8.3

$$\mathrm{res}_{z_0}(h) = a_{-1}.$$

**Lemma 8.9**: 1) If $f$ has a simple (order 1) pole at $z_0$ then

$$\mathrm{res}_{z_0}(f) = \lim_{z \to z_0} (z-z_0) f(z)$$

2) If $f$ has a pole of order $n$ then

$$\mathrm{res}_{z_0}(f) = \lim_{z \to z_0} \frac{1}{(n-1)!} \left(\frac{d}{dz}\right)^{n-1} (z-z_0)^n f(z).$$

**Proof**: $(z-z_0)^n f(z) = a_{-n} + a_{-n+1}(z-z_0) + \dots + a_{-1}(z-z_0)^{n-1} + O((z-z_0)^n)$. $\square$

## 8.ii) The Residue Formula

**Thm 8.10** Suppose $C = \partial D$ is a circle around $z_0$. Then if $h : D \to \mathbb{C}$ is meromorphic,

$$\oint_C h(z)\,dz = 2\pi i \, \mathrm{res}_{z_0}(h).$$

**Proof**: By prop 8.6,

$$h = \frac{a_{-n}}{(z-z_0)^n} + \dots + \frac{a_{-1}}{(z-z_0)} + \underbrace{a_0 + \dots}_{:= f \text{ holomorphic so } \int_C f\,dz = 0}$$

and apply observation from last time

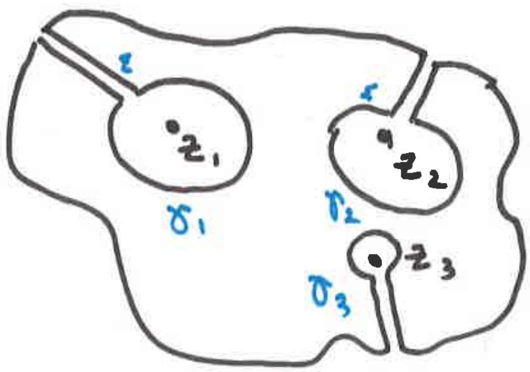
$$\int z^n dz = \begin{cases} 2\pi i & n = -1 \\ 0 & \text{else} \end{cases}.$$

$\square$

**8.11 Theorem (The Residue Theorem):** Let $\Gamma$ be a simple closed curve in $\Omega$, and $h : \Omega \to \mathbb{C}$ meromorphic, with poles $z_1, \dots, z_N$. Then

$$\oint_{\Gamma} h(z)\,dz = 2\pi i \sum_{k=1}^{N} \operatorname{res}_{z_k}(h)$$

**Proof:** Consider




show $\displaystyle\int_{/\!/} h(z)\,dz \longrightarrow 0$, so

$$0 = \lim_{\varepsilon \to 0} \underbrace{\int_{\Gamma \setminus S_\varepsilon} h(z)\,dz}_{= \oint_{\Gamma} h(z)\,dz} + \underbrace{\sum_{i=1}^{N} \int_{\gamma_i \setminus S_\varepsilon} h(z)\,dz}_{= -2\pi i \sum_{i=1}^{N} \operatorname{res}_{z_i}(h)} + \cancel{\int_{/\!/} h(z)\,dz}^{\approx 0}.$$

$\square$

**Def 8.12:** Evaluating an integral by choosing an appropriate curve/contour then calculating residues is known as <u>contour integration</u>.

# Lecture 9: Contour Integration II (with poles)

Recall: If $\Gamma$ is a closed curve in $\Omega$, and $h : \Omega \to \mathbb{C}$ meromorphic

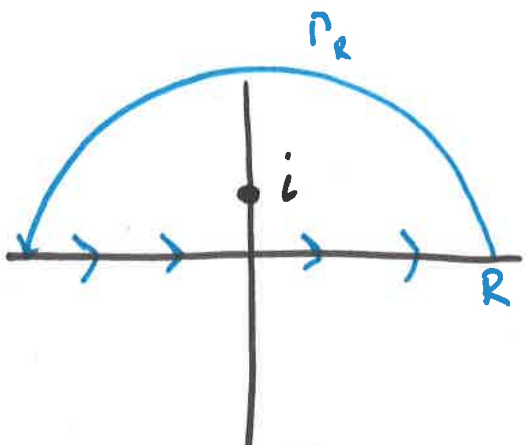
then
$$\oint_\Gamma h(z)dz = 2\pi i \sum_{k=1}^{N} \operatorname{res}_{z_k}(h)$$

where $z_1, \dots z_N$ are poles.

## 9.i) Examples of simple contour integration

**Ex 9.1:** Evaluate $\displaystyle\int_{-\infty}^{\infty} \frac{dx}{1+x^2} = I$

Set $h(z) = \dfrac{1}{1+z^2}$ poles where $0 = 1+z^2$
ie $z = \pm i$. (both simple)




$$\lim_{R\to\infty} I + \cancel{\int_{\Gamma_R} \frac{1}{1+z^2}\,dz} = 2\pi i \left[\operatorname{res}_i(h) + \cancel{\operatorname{res}_{-i}(h)}\right]$$

not enclosed

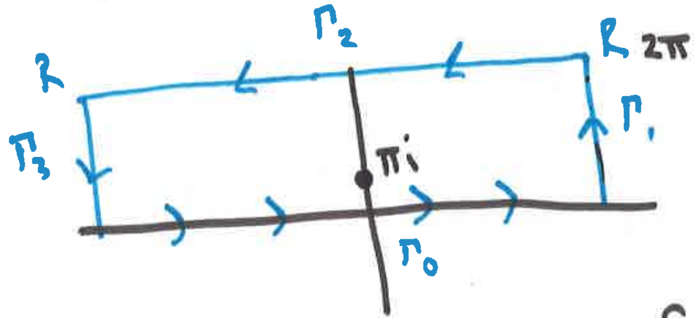
$$\le \frac{CR}{1+R^2} \to 0$$

$$= 2\pi i \left[ \left.\frac{1}{(z+i)}\right|_{z=i} \right]$$

$$= \frac{2\pi i}{2i} = \boxed{\pi}$$

Ex 9.2 : Evaluate $I = \int_{-\infty}^{\infty} \frac{e^{ax}}{1+e^x}\,dx$ for $a \in (0,1)$. $\boxed{9.2}$

Set $h(z) = \frac{e^{az}}{1+e^z}$ $\quad 1+e^z = 0 \iff z = i\pi(2k+1)$




$$\lim_{R\to\infty} \int_{\Gamma_0} h(z)dz + \ldots + \int_{\Gamma_3} h(z)dz = 2\pi i \operatorname{Res}_{i\pi}(h)$$

Careful,
$$\lim_{R\to\infty} \int_{\Gamma_0} h(z)dz = I$$
$$\lim_{R\to\infty} \int_{\Gamma_2} \frac{e^{a(x+2\pi i)}}{1+e^{(x+2\pi i)}} = -e^{2\pi i a} I$$
$$\lim_{R\to\infty} \int_{\Gamma_1} h(z)dz = \int_0^{2\pi} \left|\frac{e^{a(R+it)}}{1+e^{(R+it)}}\right| dt \leq Ce^{(a-1)R} \to 0$$
for $|a|<1$.

$$(1-e^{2\pi i a})I = 2\pi i \operatorname{Res}_{i\pi}(h)$$
$$= \lim_{z\to i\pi} e^{az}\frac{(z-\pi i)}{1+e^z}$$
$$= \lim_{z\to i\pi} e^{az}\left(\frac{z-\pi i}{e^z - e^{i\pi}}\right) = \frac{1}{\frac{d}{dz}e^z\big|_{i\pi}}$$
$$= e^{-a\pi i}$$

$$I = \frac{2\pi i}{e^{\pi i a} - e^{-\pi i a}} = \boxed{\frac{\pi}{\sin \pi a}}$$

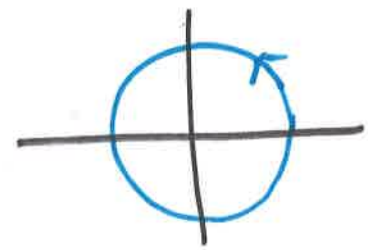
## 9.ii) Examples with Trig functions

Let $R(x,y)$ be a rational function

$$\int_0^{2\pi} R(\sin\theta, \cos\theta)\, d\theta$$

$$\cos\theta = \frac{1}{2}\left(z + \frac{1}{z}\right) \qquad \text{if } z = e^{i\theta}$$

$$\sin\theta = \frac{1}{2i}\left(z - \frac{1}{z}\right)$$



So $R(\cos, \sin)$ is meromorphic, and

$$\int_0^{2\pi} R(\cos\theta, \sin\theta)\, d\theta = \int_{S^1} R\left(\frac{1}{2i}\left(z - \frac{1}{z}\right), \frac{1}{2}\left(z + \frac{1}{z}\right)\right) \frac{dz}{iz}.$$

Ex 9.3: Evaluate for $0 < a < 1$.

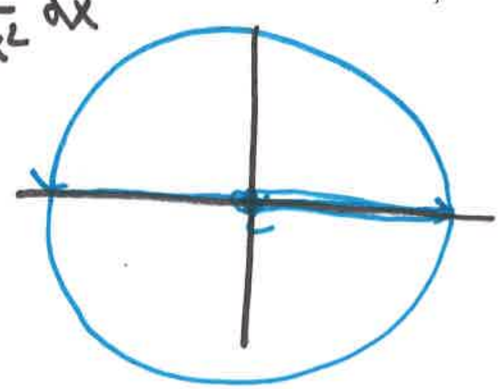
$$\int_0^{2\pi} \frac{d\theta}{1 + a^2 - 2a\cos\theta} = \int_{S^1} \frac{i\, dz}{(z-a)(az-1)} = 2\pi i\left(\frac{i}{a^2-1}\right)$$

$$= \frac{2\pi}{1-a^2}$$

## 9.iii) Fractional powers of x.

Ex 9.4: Evaluate $I(a) = \int_0^{\infty} \frac{x^a}{1+x^2}\, dx$

Take $h(z) = \frac{z^a}{1+z^2}$



# Lecture 10 | Removable + Essential Singularities

Recall meromorphic functions have three types of singularities

1) removable $\quad C : D\setminus\{0\} \to \mathbb{C}$.
2) poles $\quad (\frac{1}{z})$
3) essential singularities $(e^{-\frac{1}{z}})$

## 10.i) Removable Singularities

**Def 10.1**: Suppose $f : \Omega\setminus\{z_0\} \to \mathbb{C}$ is holomorphic. It is said to have a removable singularity at $z_0$ if $\exists \tilde{f} : \Omega \to \mathbb{C}$ holomorphic w/ $\tilde{f} = f$ on $\Omega\setminus\{z_0\}$.

**Theorem 10.2** (Riemann's removable singularity theorem)
Suppose $f : \Omega\setminus\{z_0\} \to \mathbb{C}$ is holomorphic, and $|f| < C_0$ is bounded on a disk $D(z_0, \delta)$ for some $\delta > 0$. Then $f$ has a removable singularity at $z_0$.

**Corollary 10.3**: A singularity is of type 2) if and only if $|f(z)| \to \infty$ as $z \to z_0$.

**Proof**: $\Rightarrow$ Suppose $f$ has a pole/essential sing. $f = \sum_{n=-M}^{\infty} a_n (z-z_0)^n$

so $|f(z)| \to \infty$.

$\Leftarrow$ Suppose $|f(z)| \to \infty$. Then $\lim_{z\to z_0} \frac{1}{|f(z)|} = 0$, so $\frac{1}{f(z)}$ has a removable singularity by Thm 10.2, and by continuity $\frac{1}{f(z_0)} = 0$. Therefore $\frac{1}{f}$ has a zero and is holomorphic $\Rightarrow f$ is meromorphic.

**Rem 10.4**: For essential singularities, limit does not exist.

**Rem 10.5**: $\left\{ \begin{matrix} \text{Sol. of elliptic} \\ \text{PDEs} \end{matrix} \right\} \subseteq \left\{ \begin{matrix} \text{holomorphic} \\ \text{functions} \end{matrix} \right\}$

Many other PDEs have removable singularity thms, e.g. Yang-Mills fields.

# 10.ii) Proof of removable singularity thm

**Proof** (of theorem 10.2)

Let $C \subseteq \Omega \setminus z_0$ be $C = \partial D(z_0, \delta)$.

**Step 1**: Define $\tilde{f}$ by

$$\tilde{f}(z) := \frac{1}{2\pi i} \int_C \frac{f(\zeta)}{\zeta - z} d\zeta.$$

**Step 2**: $\tilde{f}$ is holomorphic on $D(z_0, \delta)$.

$$\bar{\partial}_z \tilde{f} = \frac{1}{2\pi i} \bar{\partial}_z \int_C \frac{f(\zeta)}{\zeta - z} d\zeta$$

$$= \frac{1}{2\pi i} \int_C \bar{\partial}_z \frac{f(\zeta)}{\zeta - z} d\zeta = 0.$$

Commute derivative and integral. *

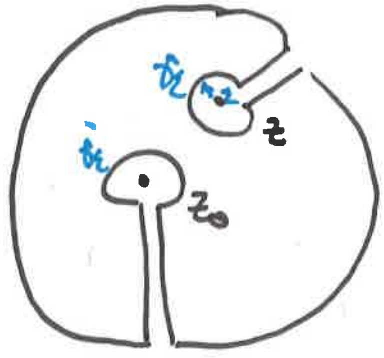
* justification $\lim_{h \to 0} \frac{1}{2\pi i} \int_C \frac{f(\zeta)}{\zeta - z + h} - \frac{f(\zeta)}{\zeta - z}$

use $\frac{1}{\zeta - z + h} = \frac{1}{(\zeta - z)(1 + \frac{h}{\zeta - z})} = \frac{1}{\zeta - z}\left(1 - \frac{h}{\zeta - z} + \frac{h^2}{(\zeta - z)^2} + \dots\right)$

$$= \lim_{h \to 0} \frac{1}{2\pi i} \int_C \frac{f(\zeta)}{(\zeta - z)^2} + h \int_C \frac{f(\zeta)}{(\zeta - z)^3} + \dots d\zeta$$

$$= \frac{1}{2\pi i} \int_C \frac{f(\zeta)}{(\zeta - z)^2} d\zeta.$$

**Step 3**: $\tilde{f} = f$ on $\partial D \setminus z_0$. By Cauchy,




$$0 = \int_{C \setminus s_\varepsilon} \frac{f(\zeta)}{\zeta - z} d\zeta + \int_{\gamma_\varepsilon} \frac{f(\zeta)}{\zeta - z} d\zeta + \int_{\gamma_\varepsilon'} \frac{f(\zeta)}{\zeta - z} d\zeta$$

$$\lim_{\varepsilon\to 0}\int_{C-s_\varepsilon}\frac{f(\zeta)}{\zeta-z}\,d\zeta=\int_C\frac{f(\zeta)}{\zeta-z}\,d\zeta$$

$$\lim_{\varepsilon\to 0}\int_{\delta_\varepsilon}\frac{f(\zeta)}{\zeta-z}\,d\zeta=-2\pi i\,f(z)$$

$$\lim_{\varepsilon\to 0}\int_{\sigma'}\frac{f(\zeta)}{(\zeta-z)}\,d\zeta=0$$ bc $\frac{|f(\zeta)|}{|\zeta-z|}<C$ if $z\neq z_0$,

so $\left|\int_{\sigma_\varepsilon'}\frac{f(\zeta)}{\zeta-z}\,d\zeta\right|\leq C\varepsilon\to 0.$ □

## 10.iii) : Essential Singularities

Corollary 10.3 suggests that for essential singularities $\lim_{z\to z_0}|f(z_0)|$ cannot exist.

The next theorem confirms this.

Thm 10.6 : (Casorati - Weierstraß) Suppose $r>0$ and $f: D_r(z_0)\setminus\{z_0\}\to\mathbb{C}$ is holomorphic with an essential singularity at $z_0$. Then $\operatorname{Im}(D_r(z_0)\setminus z_0)\subseteq\mathbb{C}$ is dense.

Proof : Suppose not. $\exists w,\delta$ with $|f(z)-w|>\delta$ for all $z\in D_r(z_0)\setminus\{z_0\}$. Consider $g(z)=\frac{1}{f(z)-w}$.

It is bounded, hence $g(z)$ is holomorphic on $D_r(z_0)-\{z_0\}$.

Either i) $g(z_0)\neq 0$ in which case $f(z)-w$ is holomorphic

$=0$ meromorphic $\to\leftarrow$ □

## 10.iv) Meromorphic functions on $\mathbb{C}$

$f : \mathbb{C} \to \mathbb{C}$ holo/meromorphic. We can use singularities to characterize behavior at infinity. Let

$$F(z) = f\left(\tfrac{1}{z}\right)$$

Then $F(z)$ is holomorphic for $\frac{1}{z}$ small as $z \to \infty$.

**Def 10.7**: A function $f$ is holomorphic / meromorphic / has an essential singularity at $\infty$ if $F(z)$ the same a $z = 0$.

**Thm 10.8**: Suppose $f : \mathbb{C} \to \mathbb{C}$ is meromorphic including at $\infty$. Then $f(z) = \frac{p(z)}{q(z)}$ for $p, q : \mathbb{C} \to \mathbb{C}$ polynomials.

**Proof**: If $f$ is meromorphic including at $\infty$ it must have finitely many poles (Taylor series $\Rightarrow$ poles are isolated) Call them $z_1 \dots z_N$.

$$f(z)\Big|_{D(z_k, r)} = \underbrace{\frac{a_k}{(z-z_k)^{m_k}} + \dots + \frac{a_{-1}}{(z-z_k)}}_{f_k} + \underbrace{g_k}_{\text{holomorphic}}$$

$$f\left(\tfrac{1}{z}\right) = \underbrace{\qquad\qquad\qquad}_{f_\infty(w) \quad w = \frac{1}{z}} + \underbrace{g_\infty}_{\text{holomorphic at } \infty.}$$

Consider $H = f - \sum f_k - f_\infty$. By subtracting $f_k$, $f - f_k$ is holomorphic so bounded at each $z_k$. Also bounded at $\infty$ by same for $f_\infty$. Hence $H = \text{Const}$. $\square$

Rem 10.9 : Note Thm 10.8 shows an equivalence between

$$\left\{\begin{array}{c}\text{complex}\\ \text{analytic data}\\ \text{on } \mathbb{C}\cup\{\infty\}\end{array}\right\} \underset{\cong}{\overset{\sim}{\longleftrightarrow}} \left\{\begin{array}{c}\text{algebraic data}\\ \text{in } \mathbb{C}[z,\tfrac{1}{z}]\end{array}\right\}.$$

(complex geometry) (algebraic geometry)

It is a general principle that in many situations <u>all</u> complex geometry comes from algebraic objects.

$\boxed{\text{Lecture 11}}$ The Argument Principle + Applications. 

The expression for $\partial_z$ in polar coordinates is

$$\partial_z = \tfrac{1}{2}e^{-i\theta}\left(\partial_r - \tfrac{i}{r}\partial_\theta\right)$$

Write $\quad f(z) = f(re^{i\theta}) = R(r,\theta)\,e^{i\Theta(r,\theta)}$.

argument/ angle of $f$.

Consider

$$\frac{f'}{f} = \frac{\tfrac{1}{2}e^{-i\theta}\left(\partial_r - \tfrac{i}{r}\partial_\theta\right)\left[Re^{i\Theta}\right]}{Re^{i\Theta}}$$

$$= \tfrac{1}{2}e^{-i\theta}\left[\frac{\partial_r R e^{i\Theta} + i\partial_r\Theta\, Re^{i\Theta}}{Re^{i\Theta}} + \{\text{same w/ } \theta\}\right]$$

$$= \tfrac{1}{2}e^{-i\theta}\left[\frac{R'}{R} + i\Theta' + \ldots\right]$$

$$= \tfrac{1}{2}e^{-i\theta}\left[\frac{\partial}{\partial z}(\ln(R)) + \frac{\partial}{\partial z}\Theta\right]$$

$\Rightarrow$ Since $R \in \mathbb{R}$, $C$ closed

$$\int_C \frac{f'}{f} = \underbrace{\cancel{\int_C \frac{\partial}{\partial z}\ln(R)}}_{=0} + \int_C \frac{\partial}{\partial z}\Theta$$

$$= 2\pi K$$

$$= \{\text{total change in argument}\}$$

<u>Thm 11.1</u> (Argument Principle) Suppose $f:\Omega \to \mathbb{C}$ is meromorphic. If $C \subseteq \Omega$ is closed and disjoint from zeros/poles then

$$\frac{1}{2\pi i}\oint_C \frac{f'(z)}{f(z)}\,dz = \#\{\text{zeros of } f\} - \#\{\text{poles of } f\}$$

counted with multiplicity.

**Proof**: Suppose $f$ has a zero of order $n$ at $z_0$.

1)
$$f(z) = (z-z_0)^n g(z)$$
$$f'(z) = n(z-z_0)^{n-1} g(z) + (z-z_0) g'(z)$$
$$\frac{f'}{f} = \frac{n}{(z-z_0)} + \underbrace{\frac{g'}{g}}_{\text{holomorphic}}$$
pole w/ residue $n$

2)
$$f(z) = (z-z_0)^{-n} h(z)$$
$$\frac{f'}{f} = \frac{-n}{z-z_0} + \underbrace{\frac{h'}{h}}_{\text{holomorphic}}$$
→ poles w/ residue $-n$.

By Cauchy Integral Formula
$$\frac{1}{2\pi i}\oint_C \frac{f'}{f}\, dz = \sum_{k=1}^{N} \operatorname{res}_{\frac{f'}{f}}(z_k) \qquad \square.$$

## 11.ii) Applications of the argument Principle

**Thm 11.2** (Rouché's Thm): $f, g : \Omega \to \mathbb{C}$ holomorphic, $C \subseteq \Omega$ closed. If $|f(z)| > |g(z)|$ for $z \in C$ then $f$, $g+f$ have the same number of zeros inside $C$.

**Proof**: Set $f_t(z) = f(z) + t g(z)$. Since $|f(z) - t g(z)| > 0$, the argument principle says
$$N_t = \frac{1}{2\pi i}\oint_C \frac{f_t'}{f_t}\, dz = \#\{\text{zeros of } f_t\}.$$
So enough to show that $N_t : [0,1] \to \overset{\mathbb{Z}}{\underset{\cap}{}}\mathbb{R}$ is continuous, but this is obvious b/c its an integral and bounded on $C$. $\square$

**Def 11.3**: A function $f : U \to V$ is said to be open if the image of open sets is open.

**Ex 11.4**: $\begin{cases} e^{-1/x} & x>0 \\ 0 & x \le 0 \end{cases}$ on $\mathbb{R}$ has image $[0,1)$ not open.

**Thm 11.5** (Open Mapping theorem)

If $f : \underset{(\text{open})}{\Omega} \to \mathbb{C}$ is holomorphic and non-constant, then $f$ is open.

**Proof**: Suppose $w \in \mathrm{Im}(f)$ $w = f(z_0)$. Suffices to show $\exists z \in \Omega$ st $f(z) = w'$ $\forall\ |w - w'| < \cancel{\delta}\varepsilon$ for some $\cancel{\delta}\varepsilon$

Set $g_w(z) = f(z) - w$.

$$= \underbrace{(f(z) - w_0)}_{F} + \underbrace{(w_0 - w)}_{G}$$

Choose $\delta > 0$ so that for $|z - z_0| \le \delta$ on has $f(z) \neq w_0$ on (zeros are isolated)

Set $\varepsilon$ so that $|f(z) - w_0| \ge \varepsilon$ on $|z - z_0| = \delta := C$.

Since $|w - w_0| < \varepsilon$ then

$$|F(z)| > |G(z)|$$

so zeros of $F = 1$

$= $ zeros of $f + G$. □

# 11.iii) Strong Maximum Principle

Thm 11.6 (Maximum principle part 2)

If $f$ is not constant and holomorphic on $\Omega$, then $f$ cannot attain a maximum in interior($\Omega$).

Proof: If $z_0 \in$ Interior($\Omega$) then $f(D_\varepsilon(z_0))$ is open
is a maximum

for $\varepsilon$ sufficiently small, and must contain points w/ $|f(z)| > |f(z_0)|$ $\rightarrow\leftarrow$. $\square$

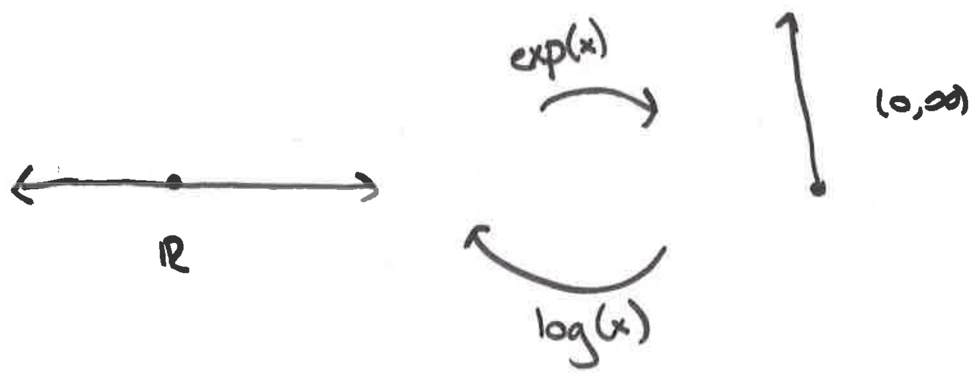
Corollary 11.7: $\sup_{\Omega} |f(z)| \leq \sup_{\partial\Omega} |f(z)|$.

provided $\Omega$ has compact closure.

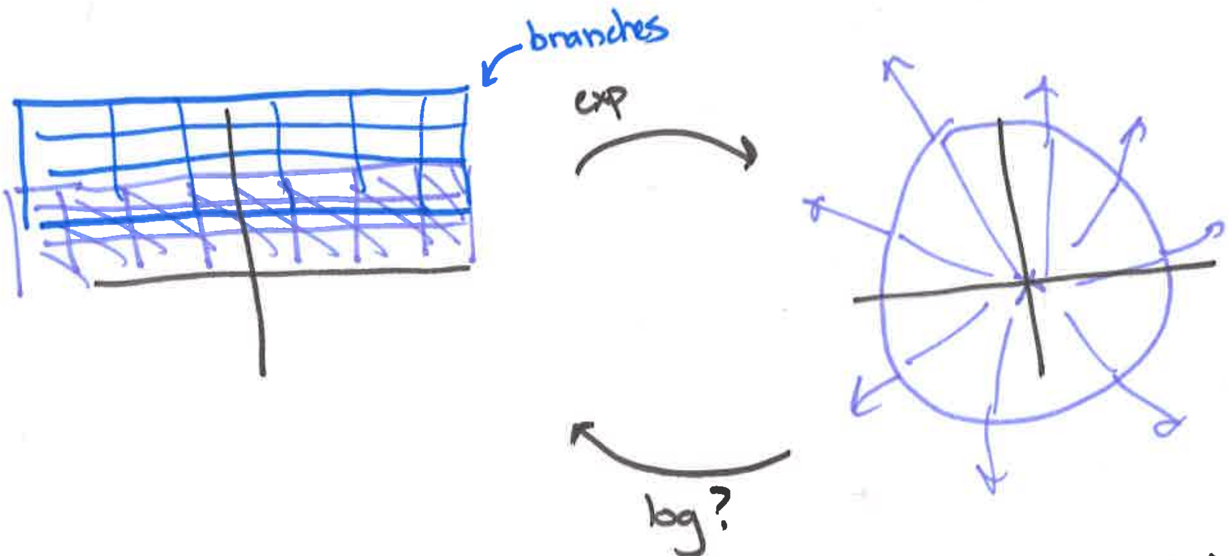
If time: Aside on linking # & of algebraic curves.

# Lecture 12 Branch Cuts and the logarithm

Recall




is a homeomorphism.




Each strip of height $2\pi$ maps bijectively to $\mathbb{C} \setminus \{0\}$. There are many (equally valid) choices of inverse!

## 12.i The Complex Logarithm

It's somewhat "natural" to use the counter clockwise angle branch:

**Prop 12.1**: Suppose $\Omega \subseteq \mathbb{C} \setminus \{0\}$ is simply connected and $1 \in \Omega$. Then $\exists\, F : \Omega \to \mathbb{C}$ $F(z) = \log_\Omega z$ a branch of logarithm such that

i) $F$ is holomorphic

ii) $e^{F(z)} = z \quad \forall z \in \Omega$

iii) $F(r) = \log(r)$ if $r \in \mathbb{R} \subseteq \mathbb{C}$.

Proof: Recall $\partial_x \log x = \frac{1}{x}$. Let $\gamma_z : [0,1] \to \Omega$ be a path from $1$ to $z$. Set

$$F(z) = \int_{\gamma} \frac{1}{\zeta}\, d\zeta .$$

Since $\Omega$ is simply connected any other choice of path has

$$F^{`}(z) - F(z) = \int_{\gamma} \frac{1}{\zeta} d\zeta - \int_{\gamma'} \frac{1}{\zeta} d\zeta = \int_{D} \bar{\partial}\left(\frac{1}{\zeta}\right) d\zeta = 0,$$

so $F$ is well-defined.

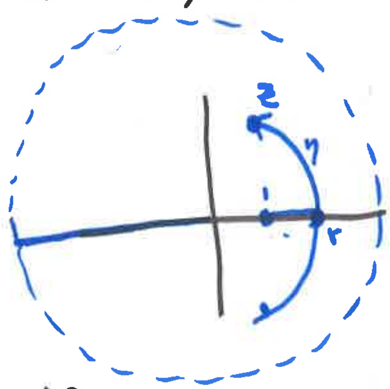
i) As before $\partial_z F(z) = \lim \frac{1}{h}\int_z^{z+h} \frac{1}{\zeta} d\zeta = \lim \frac{1}{h} h \left[\frac{1}{z}\left(1 + \frac{h}{z} + \ldots\right)\right]$
$= \frac{1}{z}$

so holomorphic.

ii) Enough to show $\partial_z \left(z e^{-F(z)}\right) = 0$ and $F(1) = 1$.

$$\frac{d}{dz}\left(z e^{-F(z)}\right) = e^{-F(z)} - zF^{`}(z) e^{-F(z)} = e^{-F(z)}\left[1 - z \cdot \frac{1}{z}\right] = 0$$

iii) If $x \in \mathbb{R}$ is near $1$, then we can choose $\gamma$ along $x$-axis, and use FTC. $\square$.

Ex 12.2: Consider $\Omega =$  $= \mathbb{C} \setminus \{(-\infty, 0]\}$.



In this case at $z = re^{i\theta}$ for $\theta \neq \pm\pi$,

$$\log re^{i\theta} = \log z = \int_1^r \frac{dx}{x} + \int_{\eta} \frac{dw}{w}$$

$$= \log r + i\theta . = \log r + \text{"}\log e^{i\theta}\text{"}$$

**Ex 12.3**: It is <u>not</u> true that
$$\log(z_1 z_2) = \log z_1 + \log z_2.$$
Take $z_1 = z_2 = e^{\frac{2\pi i}{3}}$. Then
$$\log(z_1 z_2) = \log(e^{\frac{4\pi i}{3}}) = \log(e^{-\frac{2\pi i}{3}}) = -\frac{2\pi}{3}i$$
but
$$\log(z_1) = \log(z_2) = \frac{2\pi i}{3}.$$

**Prop 12.4**: The principal branch of the logarithm has Taylor series
$$\log(1+z) = \sum_{n=0}^{\infty} (-1)^n \frac{z^n}{n}.$$
**Proof**: This holds for $z \in \mathbb{R} \setminus (-\infty, 0]$, so by analytic continuation it holds everywhere the series converges $(|z|<1)$. $\square$.

## 12.ii: Other powers and exponents

**Lemma 12.5**: Suppose $\Omega$ is a simply-connected domain in $\mathbb{C} \setminus \{0\}$, w/ $1 \in \Omega$.
Then $\forall \alpha \in \mathbb{C}$, $\exists\, z^\alpha : \Omega \to \mathbb{C}$.

**Proof**: Set
$$z^\alpha = (e^{\log z})^\alpha = e^{\alpha \log z}$$
where $\log$ is defined using a branch cut w/ $\log 1 = 0$.
This satisfies

i) $1^\alpha = 1$.

ii) $(z^{1/n})^n = e^{\frac{1}{n}\log z} \cdot \ldots \cdot e^{\frac{1}{n}\log z}$
$= e^{\log z} = z.$ $\square$

**Prop 12.6**: Suppose $f : \Omega \to \mathbb{C}$ vanishes nowhere, and $\Omega$ is simply-conn.
then $\exists g$ holomorphic satisfying
$$f(z) = e^{g(z)}.$$

**Proof:** Set

$$g(z) = \int_\gamma \frac{f'(\zeta)}{f(\zeta)} d\zeta + c_0$$

where $c_0 = f(z_0)$ and $\gamma$ is a path from $z$ to $z_0$. This is well-defined as before with

$$\partial_z g(z) = \frac{f'(z)}{f(z)}$$

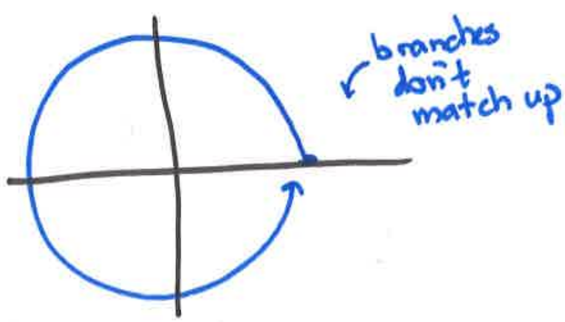
Then
$$\frac{d}{dz}\left(f(z)e^{-g(z)}\right) = f'(z)e^{-g(z)} + f(z)e^{-g(z)} \cdot \frac{-f'(z)}{f(z)}$$
$$= 0$$

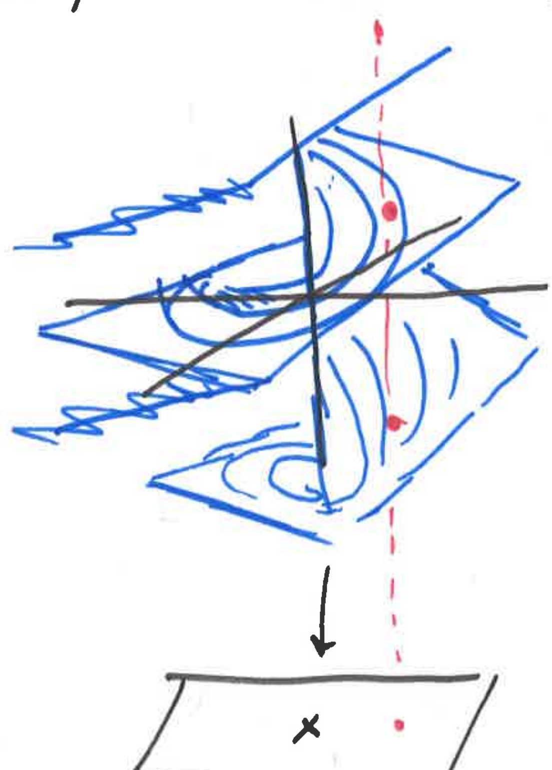
and $g(z_0) = c_0$ so $f(z_0)e^{-g(z_0)} = 1$ $\square$.

## 12.iv) Multi-Valued Functions

Another viewpoint is to allow "multi-valued functions" so

$$\log(re^{i\theta}) = \{\log r + i\theta,\ i\theta + 2\pi,\ +4\pi, \ldots\}.$$



This makes
$$\begin{array}{c} L \simeq \mathbb{C} \\ \downarrow \\ \mathbb{C}^\times \end{array}$$

into a covering map w/ deck transformations $\mathbb{Z}$ given by

$$k \to \left[(x+iy) \to (x+i(y+2\pi k))\right]$$

A multi-valued function is a section $s: \mathbb{C}^\times \longrightarrow L$ w/ $\pi \circ s = \mathrm{Id}$.

**Modern-Viewpoint:** $L$ is a Riemann surface (Math 117).

# Lecture 13 | Entire Functions I: the basic properties.

**Def 13.1**: A function is said to be entire if it is holomorphic as a map $f : \mathbb{C} \to \mathbb{C}$.

**Part III**: Study and characterize entire functions

Q1: What can the zeros $f^{-1}(0)$ look like?
(note no accumulation points by analytic continuation)

Q2: What can the growth at $|z| \to \infty$ look like
(note cannot be bounded by Liouville's theorem)

Q3: To what extent do Q1 and Q2 uniquely determine $f$?
(Hadmard Factorization Theorem)

Q4: What can be said about specific entire functions (meromorphic) and their applications in number theory, combinatorics, etc.

**Rem 13.2**: Answering Q1 in the meromorphic case includes the Riemann hypothesis (zeros of the $\zeta$-function).

## 13.i Examples of Entire Functions

**Ex 13.3**: (Standard Trig functions)

$$\cos(z) = \frac{e^{iz} + e^{-iz}}{2} = \sum_{n=0}^{\infty} (-1)^n \frac{z^{2n}}{2n!}$$

$$\sin(z) = \frac{e^{iz} - e^{-iz}}{2i} = \sum_{n=0}^{\infty} (-1)^n \frac{z^{2n+1}}{(2n+1)!}$$

**Ex 13.4** (Hyperbolic Trig functions)

$$\cosh(z) = \frac{e^{z} + e^{-z}}{2} = \cos(-iz) = \sum_{n=0}^{\infty} \frac{z^{2n}}{2n!}$$

$$\sinh(z) = \frac{e^{z} - e^{-z}}{2} = \sin(-iz)$$

**Ex 13.5**: Various compositions:

$$e^{az^2}, \quad \cos(z^2)$$

**Ex 13.6** (arbitrary growth)

$$e^{e^z}, \; e^{e^{e^z}}, \; \ldots$$

**Ex 13.7** (Special Functions)

$$\left[(r\partial_r)^2 + (r\partial_r) + (r^2 - \alpha^2)\right] J_\alpha(r) = 0$$

Solutions of the Bessel ODE for $\alpha \in \mathbb{C}$.

- $J_\alpha(z)$ is entire if $\alpha \in \mathbb{Z}$
- $J_\alpha(z)$ is entire as a function of $\alpha$ for fixed $r \in (0, \infty)$.

Rem: Appear as radial part of $\Delta = \partial_x^2 + \partial_y^2$ for heat, wave, Schrodinger, $\Rightarrow$ EM, Heat, diffusion, QFT propagators, etc.

## 13.ii Properties of Entire Functions

**Ex 13.8**: $P_f(z) = \sum_{n=0}^{\infty} a_n z^n$ is entire if $R = \infty$ (radius of convergence)

**Prop 13.9**: $P_f$ is entire if and only $\lim_{n\to\infty} |a_n|^{\frac{1}{n}} = 0$.

**Proof**: Recall that $\frac{1}{R} = \limsup |a_n|^{1/n}$. So if $\lim_{n\to\infty} = 0$ then $R = \infty$ and $P(z)$ converges everywhere.
Conversely, if $P(z)$ is entire but $\lim_{n\to\infty} |a_n|^{1/n} \neq 0$ proof of Cartan-Hadamard ("only if exercise") shows a contradiction. $\square$.

**Prop 13.10**: If $f$ is entire, then it has a convergent power series expansion with $R = \infty$.

**Proof**: The proof of existence of power series actually implies this.

$$f(z) = \left(\frac{1}{2\pi i}\oint_C \frac{f(\zeta)}{\zeta^{n+1}}\,d\zeta\right) z^n$$

for any curve $C$ such that $C = \partial D$ w/ $D \subseteq \Omega$, $z \in D$.

In this case $D = D_R$ can be taken to be any radius,

and by Cauchy

$$\oint_{C_{R_1}} \frac{f(\zeta)}{\zeta^{n+1}}\,d\zeta = \oint_{C_{R_2}} \frac{f(\zeta)}{\zeta^{n+1}}\,d\zeta.$$

□

**Thm 13.11** (Cauchy's Inequalities)

Suppose $f : \Omega \to \mathbb{C}$ is holomorphic, and $D \subseteq \Omega$ is a disk centered at $z_0$, w/ radius $R$. Then

$$|f^{(n)}(z_0)| \leq \frac{n!\,\|f\|_{C^0(D)}}{R^n} \overset{\text{max principle}}{=} \frac{n!\,\|f\|_{C^0(\partial D)}}{R^n}.$$

**Proof**: By Cauchy's Integral formula

$$|f^{(n)}(z_0)| = \left|\frac{n!}{2\pi i}\oint_{\partial D} \frac{f(\zeta)}{(\zeta - z_0)^{n+1}}\,d\zeta\right|$$

$$= \left|\frac{n!}{2\pi i}\int_0^{2\pi} \frac{f(z_0 + Re^{i\theta})}{R^{n+1} e^{i(n+1)\theta}}\,Re^{i\theta}\,d\theta\right|$$

$$\leq \frac{n!}{2\pi}\,\|f\|_{C^0(\partial D)} \cdot \frac{1}{R^n} \cdot 2\pi$$

□

**Thm 13.12**: Suppose that $f$ is entire and

$$|f(z)| \le M |z|^n$$

for $n \in \mathbb{N}$ and $|z| \ge 1$. Then $f$ is a polynomial of degree at most $n$.

**Proof**: We claim that $f^{(n)}(z)$ is bounded, hence constant by Liouville. Take $R$ large and $|z| \le R$.

$$f^{(n)}(z) = \frac{n!}{2\pi i} \int_{C = \{|w-z| = R\}} \frac{f(w)}{(w-z)^{n+1}}\, dw$$

$$\le n! \, \frac{\|f\|_C}{R^n}$$

$$\le \frac{n!}{2\pi} \frac{M |2R|^n}{R^n} < \infty \qquad \square.$$

**Thm 13.13**: Suppose $f$ is entire and

$$|f(z)| \ge m |z|^n \quad \text{once} \quad |z| > R_0 \text{ for some } R_0$$

then $f$ is necessarily a polynomial of degree $\ge n$.

**Proof**: Write $f = p(z) h(z)$ where $p(z)$ is a polynomial, $|h(z)| \neq 0$

This can be done, as the assumption implies finite zeros.

i) $\deg p = n$. is bounded, so constant.

ii) $\underbrace{\deg p}_{=d} < n$. $\frac{f(z)}{p(z)} = h(z)$ has $|h(z)| \ge |z|^{d-n}$ $\Rightarrow \frac{1}{h(z)}$ is const, $h(z)$ is const $\not\to$ so only iii)

iii) $\deg p \ge n$ $\quad \frac{z^{n-d} p(z)}{f(z)}$ is const as $|z^{n-d}||p(z)| \le |z|^n \underset{m}{\le} |f(z)|$.

# Lecture 14 | Jensen's Formula + Products

## i) Jensen's Formula

Let $f:\Omega \to \mathbb{C}$ be holomorphic w/ $f(0)\neq 0$. Denote by $D_R$, $C_R = \partial D_R$ the disk of radius $R$.

**Thm (Jensen) 14.1**: Suppose $z_1, \ldots z_N$ (w/ multiplicity) are the zeros of $f$ and $f \neq 0$ on $\partial D_R$. Then

$$* \quad \log|f(0)| = \sum_{k=1}^{N} \log\left(\frac{|z_k|}{R}\right) + \frac{1}{2\pi}\oint_{\partial D_R} \log|f(z)|\, d\theta.$$

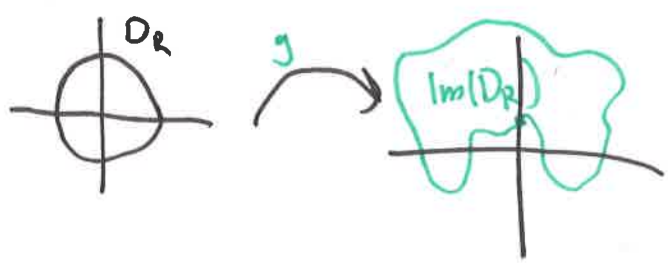
**Proof**: There are 4 Steps

**Step 1**: because $\log xy = \log x + \log y$ on $\mathbb{R}$,

$f_1, f_2$ satisfy $*$ $\Rightarrow$ $f_1 f_2$ satisfies $*$

**Step 2**: Consider $g(z) = \dfrac{f(z)}{\prod (z-z_k)}$ which vanishes nowhere and extends (via removable singularities) to be holomorphic in $\Omega$. So $f = \prod(z-z_k)\cdot g(z)$ and is suffices to prove for $g$, $z - z_k$.

**Step 3**: Since $g \neq 0$,




$\exists$ a branch of the logarithm on $Im(g)$ so set $h = \log g$

$$\log|g(z)| = \log|e^{h(z)}| = \operatorname{Re} h(z)$$

$$\operatorname{Re} h(z) = \operatorname{Re}\left[\frac{1}{2\pi i}\int \frac{h(Re^{i\theta})\, iRe^{i\theta}\, d\theta}{Re^{i\theta}}\right] = \operatorname{Re}\frac{1}{2\pi}\int_{D_R} h(z)\, d\theta = \frac{1}{2\pi}\int_{\partial D_R} \log|g(z)|\, d\theta$$

**Step 4** : Suppose $f(z) = z - w$, need

$$\log|w| = \log\left(\frac{|w|}{R}\right) + \frac{1}{2\pi}\int_0^{2\pi} \log|Re^{i\theta} - w|\, d\theta$$

Note $\log\left|\frac{w}{R}\right| = \log|w| - \log R$

$\log|Re^{i\theta} - w| = \log R + \log\left|e^{i\theta} - \frac{w}{R}\right|$.

$$\Rightarrow \log|w| = \log|w| + \log R - \log R$$
$$= \log\left|\frac{w}{R}\right| + \log R$$
$$= \log\left|\frac{w}{R}\right| + \log R + \log\left|e^{i\theta} - \frac{w}{R}\right| \quad \checkmark.$$

**Claim** : $\int_0^{2\pi} \log|e^{i\theta} - a|\, d\theta = 0$ for $|a| = \left|\frac{w}{R}\right| < 1$.

**Proof** : As in step 2, $F(z) = 1 - az$ vanishes nowhere so

$|F| = e^{\operatorname{Re} G}$ w/ $\log F = G$.

By step 2,

$$0 = \log|F(0)| = \frac{1}{2\pi}\int_{\partial D_R} \log|F(z)| \qquad \square$$

14.ii) Nevanlina's First Fundamental Thm (baby version)

Let $n(R) : \mathbb{R}^{\geq 0} \to \mathbb{N}$ denote the # of zeros inside $D_R$.

**Lemma 14.3** : $\int_0^R \frac{n(r)}{r}\, dr = \sum_{k=1}^N \log\left|\frac{R}{z_k}\right|$.

**Proof** : $\sum_{k=1}^N \log\left|\frac{R}{z_k}\right| = \sum_{k=1}^N \int_{|z_k|}^R \frac{dr}{r}$ by FTOC

$$= \sum_{k=1}^N \int_0^R \eta_k \frac{dr}{r} \quad \text{for } \eta_k = \begin{cases} 1 & R \geq |z_k| \\ 0 & \text{else} \end{cases}$$

$$= \int_0^R \frac{n(r)}{r}\, dr \qquad \square$$

**Corollary 14.4** : If $f(0) \neq 0$ and $f$ has no zeros on $\partial D_R$, then
$$\int_0^R \frac{n(r)}{r}\, dr = \frac{1}{2\pi}\int_0^{2\pi} \log|f(Re^{i\theta})|\, d\theta - \log|f(0)|.$$

**Proof** : Previous lemma + Jensen's formula $\square$

**Rem 14.5** : Note that this gives a fundamental relation/constraint relating
$$\{\#\text{ zeros of } f\} \longleftrightarrow \left\{\begin{matrix}\text{growth of } f \text{ as} \\ |z| \to \infty\end{matrix}\right\}.$$

The more sophisticated and quantitative study of this (for meromorphic functions) is Nevanlinna theory used in various areas of complex geometry + dynamics.

**Def 14.6** : $f$ has order of growth $\rho$ if
$$\rho = \inf_{\sigma}\left\{ |f(z)| \leq A e^{B|z|^{\sigma}} \text{ for } A, B \in \mathbb{R}\right\}.$$

**Lemma 14.7** : If $f$ is entire w/ growth order $\leq \rho$ then

i) $n(r) \leq C r^{\rho}$ for some $C$ and $r \geq r_0$.

ii) For $s > \rho$, $\displaystyle\sum_{k=1}^{\infty} \frac{1}{|z_k|^s} < \infty$ where $z_k$ are zeros of $f$.

**Proof** : Since $n$ is increasing,
$$\begin{aligned} n(R) &\leq \frac{1}{\sqrt{2}}\, n(R)\int_R^{2R} \frac{dr}{r} \\ &\leq C\int_0^{2R} \frac{n(r)}{r}\, dr + \log(F(0)) \\ &\leq C\int_0^{2\pi} \log|f(Re^{i\theta})|\, d\theta \leq C R^{\rho}. \end{aligned}$$
$\square$

ii) exercise

# Lecture 15 Weierstrauß Products

## 15.i) General Products

For $a_n \in \mathbb{C}$, we may consider infinite products

$$\prod_{k=1}^{\infty}(1+a_k) = \lim_{N\to\infty}\prod_{k=1}^{N}(1+a_k).$$

**Lemma 15.1**: The product $\prod_{k=1}^{\infty}(1+a_k)$ is finite if and only if $\sum_{k=1}^{\infty}|a_k| < \infty$.

**Proof**:

$$\left|\lim_{N\to\infty}\prod_{k=1}^{N}(1+a_k)\right| = \left|\lim_{N\to\infty}\prod_{k=1}^{N} e^{\log(1+a_k)}\right|$$

$$= \left|\lim_{N\to\infty} e^{\sum_{k=1}^{N}\log(1+a_k)}\right|$$

$$\le e^{\left|\lim_{N\to\infty}\sum_{k=1}^{N}\log(1+a_k)\right|}$$

$$\le e^{C\left|\sum_{k=1}^{\infty}|a_k|\right|} < \infty \quad \text{Taylor's Theorem}$$

$\square$

**Lemma 15.2**: Suppose that $f_n : \Omega \to \mathbb{C}$ are a sequence of holomorphic functions such that $\exists C_n$ w/

$$\sum_{n=0}^{\infty} C_n < \infty \quad \text{and} \quad |f_n(z) - 1| \le C_n \quad \forall z \in \Omega.$$

Then i) $f(z) := \prod_{k=1}^{\infty} f_k(z)$ converges uniformly and is holomorphic.

ii) If $f_n(z)$ doesn't vanish for any $n$,

$$\frac{f'(z)}{f(z)} = \sum_{n=1}^{\infty}\frac{f_n'(z)}{f_n(z)}$$

**Proof**: $\prod_{k=1}^{\infty} f_k(z) = \prod_{k=1}^{\infty}\left(1+(f_k(z)-1)\right)$ and $\sum_{n=1}^{\infty}|f_n(z)-1| \le \sum_{n=1}^{\infty} C_n < \infty$,

so convergence follows from previous lemma.

For holomorphicity, note that for $\gamma$ enclosing a disk

$$\oint_\gamma f(z) = \lim_{N\to\infty} \oint_\gamma f_N(z)$$
$$= 0 \quad \text{by Cauchy Integral formula.}$$

For ii) For each finite $N$,

$$\frac{\partial_z\left(\prod^N f_n(z)\right)}{\prod^N f_n(z)} = \frac{f_1'(z)f_2(z)\cdots\cdots + f_1(z)f_2'(z)\cdots}{}$$
$$= \frac{f_1'(z)}{f_1(z)} + \frac{f_2'(z)}{f_2(z)} + \cdots$$

Since $\prod^N f_n(z) \neq 0$ on some $K \subset\subset \Omega$, $\frac{f'(z)}{f(z)}$ converges there

and $\frac{f_N'(z)}{f_N(z)} \longrightarrow \frac{f'(z)}{f(z)}$. (Here we use if $g_K \to g$ and are holomorphic $g_K' \to g'$. This follows from Cauchy) $\square$

## 15.ii: The Infinite Product for Sin

**Prop 15.3:** $$\frac{\sin(\pi z)}{\pi} = z\prod_{n=1}^{\infty}\left(1 - \frac{z^2}{n^2}\right)$$

**Lemma 15.4:** $$\pi\cot(\pi z) = \sum_{n=-\infty}^{\infty}\frac{1}{z+n} = \frac{1}{z} + \sum_{n=1}^{\infty}\frac{2z}{z^2-n^2}.$$

**Proof:** First note the double sided sum means

$$\lim_{N\to\infty}\sum_{|n|\leq N}\frac{1}{z+n}$$

Note $F(z) = F(z+1)$ for $z \notin \mathbb{Z}$, and $F(z) = \frac{1}{z-n} + g(z)$ has simple pole at each $n \in \mathbb{Z}$.

Now let $\Delta(z) = \sum_{n=-\infty}^{\infty} \frac{1}{z+n}$, and consider

$$F(z) - \Delta(z) \qquad \text{for} \quad F(z) = \pi \cot(\pi z).$$

By construction, $F(z) - \Delta(z)$ has removable singularities at $n \in \mathbb{Z}$, because $\Delta$ also has simple poles at each $n \in \mathbb{Z}$.

It follows that $F(z) - \Delta(z)$ may be extended to an entire function

$$H(z) : \mathbb{C} \to \mathbb{C} \quad w/ \quad H(z+1) = H(z)$$

It then suffices to show that $H$ is bounded, by Liouville. (for $|\mathrm{Re}(z)| \leq \frac{1}{2}$)

For $|\mathrm{Im}(z)| > 1$, $z = x+iy$

$$\cot(\pi z) = i \frac{e^{-2\pi y} + e^{-2\pi i x}}{e^{-2\pi y} - e^{-2\pi i x}} < \text{const as } |y| \to \infty.$$

And for $|y|>1$

$$\left| \frac{1}{z} + \sum_{n=1}^{\infty} \frac{2z}{z^2 - n^2} \right| \leq C + \left| \sum_{n=1}^{\infty} \frac{2(x+iy)}{x^2 - (y^2+n^2) + 2ixy} \right|$$

$$\leq C + C \sum_{n=1}^{\infty} \frac{y}{y^2+n^2} < \infty$$

(Integral test)

Therefore $H(z)$ is constant. Since

$$H(z) = -H(-z) \text{ is odd, it must be } 0. \qquad \square$$

Now $G(z) = \frac{\sin \pi z}{\pi}$ $\qquad P(z) = z \prod_{n=1}^{\infty} \left(1 - \frac{z^2}{n^2}\right)$ (converges)

$$\frac{P'(z)}{P(z)} = \frac{1}{z} + \sum_{n=1}^{\infty} \frac{\frac{-2z}{n^2}}{1 - \frac{z^2}{n^2}} = \frac{1}{z} + \sum_{n=1}^{\infty} \frac{2z}{z^2 - n^2}.$$

Therefore since

$$\frac{G'(z)}{G(z)} = \pi\cot(\pi z)$$

$$\partial_z\left(\frac{P(z)}{G(z)}\right) = \frac{P(z)}{G(z)}\underbrace{\left[\frac{P'(z)}{P(z)} - \frac{G'(z)}{G(z)}\right]}_{=0}$$

$$= 0$$

so $P(z) = \text{const} \cdot \pi\cot(\pi z)$, but

$$\lim_{z\to 0}\frac{P(z)}{z} = 1 = \lim_{z\to 0}\frac{\pi}{z}\cot(\pi z) \quad \text{so const} = 1. \qquad \triangleright$$

## 15.iii : General Weierstrass Products

Def 15.5

Define the canonical factor of weight/degree $K$ by

$$E_0(z) = (1-z) \qquad E_K(z) = (1-z)\,e^{z + z^2/2 + \dots + z^K/K}.$$

Def 15.6 : Define the Weierstrauss product of a sequence $\{a_n\}$ w/ $|a_n| \to \infty$ by

$$W(z) = z^m \prod_{n=1}^{\infty} E_n(z/a_n)$$

where $m = \#\{n \mid a_n = 0\}$.

Theorem 15.7 : Given $\{a_n\}$ w/ $|a_n| \to \infty$, $\exists f : \mathbb{C} \to \mathbb{C}$ entire with $f$ vanishing at $a_n$ $\forall n$ and nowhere else. Any other such $f$ has the form $f(z)e^{g(z)}$ $g(z)$ entire.

Proof (next time) take $f = W(z)$.

# Lecture 16: Hadamard's Factorization Theorem

Recall

**Thm 15.7** For $\{a_n\} \in \mathbb{C}$ w/ $|a_n| \to \infty$, $\exists f : \mathbb{C} \to \mathbb{C}$ entire w/ zeros at $a_n$ and nowhere else, and any other such function is $f(z)e^{g(z)}$ for $g$ entire.

**Proof** (of uniqueness)

Step 1: Suppose $f_1, f_2$ are two functions satisfying the conclusions. Then $\frac{f_1}{f_2}$ is bounded near each $a_n$ $\Rightarrow$ extends holomorphically (removable sing) to $\mathbb{C}$.

and $\frac{f_1}{f_2}$ vanishes nowhere, $\Rightarrow \exists \log$ on $\operatorname{Im}\left(\frac{f_1}{f_2}\right)$ (simply connected)

$$\left( g = \int_0 \frac{f'}{f}\,dz + c_0 \right)$$

$$\frac{f_1}{f_2} = e^{g(z)}.$$

Recall $E_k(z) = (1-z)e^{z + z^2/2 + \dots + z^k/k}$  $\quad f = W(z) = z^m \prod_{n=1}^{\infty} E_n(z/a_n)$

**Step 2**: If $|z| \leq \frac{1}{2}$ $\quad |1 - E_k(z)| \leq C|z|^{k+1}$ for some $C$.

**Proof**: $(1-z) = e^{\log(1-z)}$

$$E_k(z) = e^{\log(1-z) + z + z^2 + \dots + z^k}$$

$$= e^{-z - \frac{z^2}{2} - \dots + z + z^2/2 + z^k/k}$$

$$= e^{-z^{k+1}/(k+1) - z^{k+2}/(k+2) \dots}$$

Now Taylors thm $|1 - e^{-\sim}| \leq |\sim| \leq C|z|^{k+1}$. $\square$

**Step 3**: Let $R > 0$. We will prove $f|_{D_R}$ has the necessary zeros, then $R \to \infty$.

Either $|a_n| < 2R$ (finitely many)

or $|a_n| \geq 2R$. $\Rightarrow |1 - E_n(z/a_n)| \leq c\left|\frac{z}{a_n}\right|^{n+1} \leq \frac{1}{2^{n+1}}$ const

and so $\prod_{a_n \geq 2R} E_n(z/a_n)$ converges $\square$

## 16.ii) Hadamard's Factorization Thm

Recall $f : \mathbb{C} \to \mathbb{C}$ has order of growth $\rho$ if

$$|f(z)| \le A e^{B|z|^{\rho}}$$

**Thm 16.1** (Hadamard)
Suppose $f$ is entire with order of growth $\rho$. Let $k$ be st $k \le \rho \le k+1$.
If $\{a_n\} = f^{-1}(0)$, then

$$f(z) = e^{P(z)} z^m \prod_{n=1}^{\infty} E_k(z/a_n)$$

where $P(z)$ is a polynomial of degrees $\le k$.

**Rem 16.2**: Woah! There aren't many entire functions that aren't the canonical Weierstrauss product determined by their zeros.

**Proof**:

**Lemma 16.3 (1)**:
$$|E_k(z)| \ge e^{-c|z|^{k+1}} \quad \text{if } |z| \le 1/2$$
$$\ge |1-z| e^{-c|z|^{k}} \quad \text{if } |z| \ge \tfrac{1}{2}$$

**Proof**: expand + Taylors thm as before.

**Lemma 2 (16.4)**: For any $s$ w/ $\rho < s < k+1$,

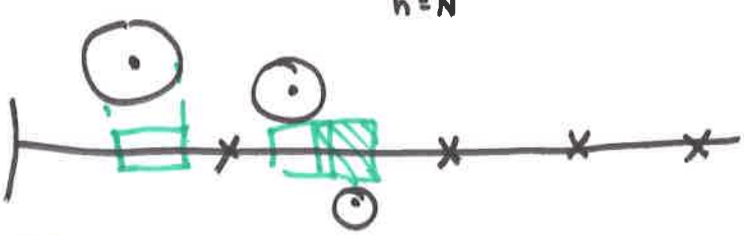
$$\left| \prod_{n=1}^{\infty} E_k(z/a_n) \right| \ge e^{-c|z|^{s}}.$$

except if $z \in B_{|a_n|^{-k-1}}(a_n)$.

**Lemma 3 (16.5)**: $\exists\, r_m \to \infty$ st

$$\left| \prod_{n=1}^{\infty} E_k\left(\frac{z}{a_n}\right) \right| \ge e^{-c|z|^{s}}.$$

**Proof**: Take $N$ st $\sum_{n=N}^{\infty} |a_n|^{-k-1} < \frac{1}{10}$.



$\sum$ |—| $< \frac{2}{10}$ so can find sequence of $r_m$ so that $|z| = r_m$ does not $\cap$ disks.

Proof of thm: By Weierstrass theorem,

$$\frac{f(z)}{E(z)} = e^{g(z)}$$

For $g(z)$ entire. So enough to show $g$ is a polynomial.

$\forall\ \rho_0 < s < k+1$

$$|f(z)| \leq C e^{B|z|^{\rho_0}}$$
$$\leq C e^{(B|z|^{\rho_0 - s}) |z|^s}$$
$$\left|\frac{f(z)}{E(z)}\right| \leq C e^{c|z|^s} \quad \text{for } |z| = r_m \text{ large enough}$$

$\Rightarrow$

$$|e^{g(z)}| \leq C e^{c|z|^s} \Rightarrow |g(z)| \leq C|z|^s \quad \text{for } |z| = r_m \to \infty.$$

Apply Theorem 13.12

$\square$.

16.iii) Proof of Lemma 2

$$\prod_{n=1}^{\infty} E_k(z/a_n) = \underbrace{\prod_{|a_n| \leq 2|z|} E_k(z/a_n)}_{①} \underbrace{\prod_{|a_n| > 2|z|} E_k(z/a_n)}_{②}.$$

②
$$\left|\prod_{|a_n| > 2|z|} E_k(z/a_n)\right| = \prod_{|a_n| > 2|z|} |E_k(z/a_n)|$$
$$\geq \prod e^{-c|z/a_n|^{k+1}} \quad \text{by Lemma 1}$$
$$\geq e^{-c|z|^{k+1}} e^{\sum \frac{c}{|a_n|^{k+1}}}$$

and if $f$ has growth order $\leq k+1$, $\underbrace{\text{then } \sum |a_n|^{-k-1} < \infty}$

$$\geq e^{-c|z|^s}.$$

Skipping proof of this, see Thm 2.1 on page 138.

①
$$\left|\prod_{|a_n| \leq 2|z|} E_k\right| \geq \prod_{|a_n| \leq 2|z|} \left|1 - \frac{z}{a_n}\right| \prod_{|a_n| \leq 2z} e^{-c\left|\frac{z}{a_n}\right|^k} \quad \text{by Lemma 1}$$

$$|a_n|^{-k} = |a_n|^{-s} |a_n|^{s-k} \leq C |a_n|^{-s} |z|^{s-k}$$

same argument as ②

$$\geq e^{-c|z|^{\rho}}$$

for second term.

$$\prod_{|a_n| \leq 2|z|} \left|1 - \frac{z}{a_n}\right| = \left|\prod \frac{a_n - z}{a_n}\right|$$

$z$ not in radius $|a_n|^{-k-1}$ around $a_k$

$$\geq \prod |a_n|^{-k-1} |a_n|^{-1} \quad \text{if } |a_n - z| > |a_n|^{-k-1}$$

$$\geq \prod |a_n|^{-k-2}.$$

And $\displaystyle \log \prod = (k+2) \sum_{2|z| > |a_n|}' \log(a_n) \leq (k+2) \,\#\text{zeros} \log 2|z|$

$$\leq |z|^{\rho_0} \log 2|z|$$

also in Thm 2.1 pg 139

$$\leq C|z|^{\rho_0 + s} \quad \forall s.$$

## 16.iv: The Little Picard Theorem

**Ex 16.7**: Suppose $f: \mathbb{C} \to \mathbb{C}$ is entire and of finite growth then $f$ is constant OR omits at most 2 values.

**Proof**: Suppose $\alpha \neq \beta$ and $f: \mathbb{C} \to \mathbb{C} - \{\alpha, \beta\}$.

$f - \alpha = e^{g(z)}$ b/c $f$ vanishes nowhere

$= e^{p(z)}$ polynomial

$f - \beta = e^{q(z)}$

$$e^{p(z)} = e^{q(z)} + (\beta - \alpha)$$

but if $q$ non constant $q(z) - (\beta - \alpha)$ has a root by FTOA, $\rightarrow\leftarrow$ b/c $e^p \neq 0$.

**Theorem 16.8** (Picard's Little Theorem): An entire function on $\mathbb{C}$ attains every value w/ at most 1 exception, unless its constant.

(Great Picard Theorem)

$\boxed{16.5}$

$\boxed{\text{Theorem}}$: If $f : \underset{\text{open}}{\Omega} \to \mathbb{C}$ has an essential singularity at $z_0$, then $f$ attains every value in $\mathbb{C}$ with at most 1 exception infinitely often on any neighborhood of $z_0$.

(Note "infinitely often" follows from "any neighborhood").

# Lecture 17 | Analytic Continuation I: the $\Gamma$-function.

Recall: If $f = g$ on a set $K \subset \Omega$ with an accumulation point, then

$$f \equiv g$$

on $\Omega$ (for $f, g$ holomorphic)

**Corollary 17.1**: Suppose $F : \Omega_1 \to \mathbb{C}$ is holomorphic, and $\Omega_1 \subset \Omega_2$. Then there is at most one $\bar{F} : \Omega_2 \to \mathbb{C}$ such that

$$\bar{F}|_{\Omega_1} = F.$$

## 17.i: The $\Gamma$-function

**Def 17.2**: For $s > 0$ in $\mathbb{R}$, define

$$\Gamma(s) = \int_0^\infty e^{-t} t^{s-1} \, dt. \quad *$$

**Lemma 17.3**: $\Gamma$ extends the function $n! : \mathbb{N} \to \mathbb{N}$ to $s > 0$ in the sense that

$$\Gamma(n+1) = n!$$

**Proof**:

$$\Gamma(n+1) = \int_0^\infty e^{-t} t^n \, dt$$

$$= (-1)^n \left(\frac{d}{dk}\right)^n \int_0^\infty e^{-kt} \, dt$$

$$= (-1)^n \left(\frac{d}{dk}\right)^n \left[-\frac{1}{k} e^{-kt}\right]_0^\infty$$

$$= (-1)^n \left(\frac{d}{dk}\right)^n k^{-1}$$

$$= (-1)^{n-1} \left(\frac{d}{dk}\right)^{n-1} \cdot 1 \cdot k^{-2} = \dots = n!. \quad \square$$

**Def/Lemma 17.4**: $\Gamma$ extends to a holomorphic function

$$\Gamma : \{\mathrm{Re}(z) > 0\} \to \mathbb{C}$$

where it is still given by $*$.

<u>Proof</u>: Recall that holomorphicity is local, so it suffices to prove this on $S_{a,b} = \{a < Re(z) < b\}$.

Also, recall if $F_n \to F$ converge uniformly and $F_n$ is holomorphic, then so is $F$. Set

$$F_n = \int_{1/n}^{n} e^{-t} t^{z-1} \, dt$$

Note that 1) $F_n$ ~~converges~~ (is finite) for each $n$, as

$$|e^{-t} t^{z-1}| \leq |e^{-t} e^{(\ln t)(z-1)}|$$

$$\leq |e^{-t} t^{(Re z - 1)}|$$

2) and ~~so~~ $F_n \to F$ pointwise, ~~so~~ since the limit in $n$ converges for fixed $z$.

3) $F_n$ is holomorphic

<u>Claim</u>: Convergence is uniform on $S_{a,b}$ and $F = \Gamma$.

$$|\Gamma(z) - F_n(z)| \leq \underbrace{\int_0^{1/n} e^{-t} t^{Re z - 1} \, dt}_{1} + \underbrace{\int_n^{\infty} e^{-t} t^{Re z - 1} \, dt}_{2}$$

1 $\leq \int |e^{-t} t^{a-1}| dt$

$\leq \int_0^{1/n} |t^{a-1}| \, dt$

$< \infty$ since $a < 1$.

2 $\leq \int_n^{\infty} e^{-t} t^{b-1} \, dt$

$< \infty$.

$\square$

## <u>17.ii)</u> Recurrence and Meromorphic extension

<u>Lemma 17.5</u>: For $Re(z) > 0$, $\Gamma$ satisfies

$$\Gamma(z+1) = z\Gamma(z).$$

<u>Proof</u>: $$0 = \int_0^{\infty} \frac{d}{dt}(e^{-t} t^s) \, dt$$ (by FTOC since integrand $\to 0$ at boundaries)

$$= s\int_0^{\infty} e^{-t} t^{s-1} \, dt + \int_0^{\infty} -e^{-t} t^s \, dt$$

$\square$

Let $\Omega_\Gamma = \mathbb{C} \setminus \{0, -1, -2, \ldots\}$.

Thm 17.6: There exists a meromorphic extension
$$\Gamma : \Omega_\Gamma \to \mathbb{C}$$
such that i) $\Gamma = ✱$ on $\{Re(z) > 0\}$
ii) $\Gamma$ has simple poles w/ residue
$$res_{-n} = \frac{(-1)^n}{n!}$$

Proof: For $-1 < Re(z) < 0$ define
$$\Gamma(z) = \frac{\Gamma(z+1)}{z}$$
It is obvious that this is meromorphic w/ a simple pole of reside $\Gamma(0) \approx \frac{\Gamma(1)}{z} = \frac{1}{z} + O(1)$
$$\Rightarrow res_0 = 1$$
at $z = 0$.

Proceeding inductively,
$$\Gamma(z) = \frac{\Gamma(z+1)}{z}$$
$\forall\ -m-1 < Re(z) < -m$ for $m \in \mathbb{N}$.
$$\Gamma(-1) = \frac{\Gamma(z+2)}{z(z+1)} \approx \frac{\Gamma(1)}{-1} \cdot \frac{1}{z+1} + O(1)$$
$$\Gamma(-2) = \frac{\Gamma(z+3)}{z(z+1)(z+2)} = \frac{\Gamma(1)}{-2 \cdot -1} \frac{1}{z+2} = \frac{1}{2!} \frac{1}{z+2} + O(1)$$
$\square$

Rem 17.7: The recurrance
$$\Gamma(z) = \frac{\Gamma(z+1)}{z}$$
continues to hold by construction.

Thm 17.8: $\Gamma(z)\Gamma(1-z) = \frac{\pi}{\sin \pi z}$ as meromorphic functions on $\mathbb{C}$.

Proof: Both are meromorphic w/ poles on $\mathbb{C} \setminus \Omega_\Gamma$, so suffices to show equality for $0 < s < 1$ in $\mathbb{R}$.

**Lemma 17.9** : For $0<s<1$

$$\int_0^\infty \frac{x^{a-1}}{1+x} = \frac{\pi}{\sin \pi a}$$

**Proof** : Contour integration. Exercise or review example from lecture 9

Use change of variables

$$\int_0^\infty \frac{x^{a-1}}{1+x}dx = \int_{-\infty}^\infty \frac{e^{ay}}{1+e^y}\,dy. \qquad \square$$

**Proof** (of Thm 17.8) :

$$\begin{aligned}
\Gamma(1-s)\Gamma(s) &= \int_0^\infty e^{-t} t^{s-1}\,\Gamma(1-s)\,dt \\
&= \int_0^\infty e^{-t} t^{s-1} \int_0^\infty e^{-u} u^{-s}\,du\,dt \quad \Big\downarrow u=vt \\
&= \int_0^\infty e^{-t} t^{s-1} \int_0^\infty t\cdot e^{-vt} (vt)^{-s}\,dv\,dt \\
&= \int_0^\infty \int_0^\infty e^{-t(v+1)} v^{-s}\,dv\,dt \quad \Big\downarrow \text{lemma 17.9} \\
&= \int_0^\infty \frac{v^{-s}}{1+v}dv = \frac{\pi}{\sin \pi(1-s)} \\
&= \frac{\pi}{\sin \pi s}. \qquad \square
\end{aligned}$$

**Corollary 17.10** : $\Gamma(\tfrac{1}{2}) = \sqrt{\pi}$.

**Proof** : $\Gamma(\tfrac{1}{2})^2 = \dfrac{\pi}{\sin(\frac{\pi}{2})}$ $\square$.

## 17.iii) Properties of $\Gamma^{-1}$.

**Thm 17.11** : $\frac{1}{\Gamma(z)}$ is an entire function with simple zeros at $z = 0, -1, -2, \ldots$

**Proof** :

$$\frac{1}{\Gamma(z)} = \underbrace{\Gamma(1-z)}_{\text{simple poles at } z=1,2,3,\ldots} \frac{\sin \pi s}{\pi} \longleftarrow \text{simple zeros at } \mathbb{Z}.$$

$\square$

**Thm 17.12** $\frac{1}{\Gamma(z)}$ has growth

$$\left|\frac{1}{\Gamma(z)}\right| \leq C e^{c|z|\log|z|}.$$

and for $\gamma = \lim_{N \to \infty} \sum_{n=1}^{N} \frac{1}{n} - \log N = 0.57721\ldots$

the Hadamard product is

$$\frac{1}{\Gamma(z)} = e^{\gamma z} \cdot z \prod^{\infty} \left(1 + \frac{z}{n}\right) e^{-z/n}.$$

$\square$

**Proof** (Stein Shakarchi pgs 165 - $\overset{K\,21}{167}$).

**Thm 17.13** (Legendre Recurrence)

$$\Gamma(z)\Gamma(z + 1/2) = 2^{1-2z}\sqrt{\pi}\,\Gamma(2z).$$

$\square$

# Lecture 18 Analytic Continuation II : the Riemann $\zeta$-function.



**Def 18.1** : For $s>1$ in $\mathbb{R}$, define
$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}.$$

Note this is convergent by the integral test.

## 18.i) Specific Values of $\zeta$

**Prop 18.2** : $\zeta(2) = \frac{\pi^2}{6}$.

**Proof** : Recall
$$\begin{aligned}
\frac{\sin(\pi z)}{\pi} &= z\prod_{n=1}^{\infty}\left(1-\frac{z^2}{n^2}\right) \\
&= z\left(1-\frac{z^2}{1}\right)\left(1-\frac{z^2}{4}\right)\left(1-\frac{z^2}{9}\right)\cdots \\
&= z + z^3\left[-1-\frac{1}{4}-\frac{1}{9}-\cdots\right] + z^5 + \cdots \\
&= z - z^3\zeta(2) + O(z^5)
\end{aligned}$$

and
$$\frac{\sin(\pi z)}{\pi} = \frac{1}{\pi}\left[\pi z - \frac{(\pi z)^3}{3!} + \frac{(\pi z)^5}{5!}\right] + \cdots$$

comparing $z^3$ coefficients
$$-z^3\zeta(2) = -\frac{\pi^2 z^3}{6} \qquad \square$$

**Prop 18.3** : $\zeta(4) = \frac{\pi^4}{90}$

**Proof** : By Fourier series on $[-\pi,\pi]$
$$\begin{aligned}
x^2 &= \frac{2}{\pi}\left[\int_0^{\pi} x^2\,dx + \sum_{n=0}^{\infty}\int_0^{\pi} x^2\cos(nx)\,dx\right] \\
&= \frac{\pi^2}{3} + \sum_{n=1}^{\infty}\frac{(-1)^n 4}{n^2}\cos(nx)
\end{aligned}$$

By Parseval's theorem,

$$\|f\|^2_{L^2[-\pi,\pi]} \simeq \|\mathcal{F}(f)\|_{\ell^2} \quad \text{(with some } \pi\text{s and } 2\text{s)}$$

$$\frac{2}{\pi}\int_{-\pi}^{\pi} x^4\,dx = 2\left(\frac{\pi^2}{3}\right)^2 + \sum_{n=1}^{\infty}\left(\frac{(-1)^n 4}{n^2}\right)^2$$

$$\frac{2\pi^4}{5} = \frac{2\pi^4}{9} + 16\zeta(4)$$

$$\Rightarrow \frac{8\pi^4}{45} = 16\zeta(4) \qquad \square$$

## 18.ii) Analytic Continuation of $\zeta(z)$.

**Def 18.4**: $\zeta(z) = \sum_{n=1}^{\infty} \frac{1}{n^z}$ converges absolutely for $\mathrm{Re}\, z > 1$.

$$\left(\left|\frac{1}{n^z}\right| = \left|e^{-z\log n}\right| = n^{-\mathrm{Re}\, z}.\right)$$

**Theorem 18.5** (Riemann) The $\zeta$ function satisfies

$$\zeta(1-s) = 2(2\pi)^{-s}\Gamma(s)\cos\left(\frac{\pi s}{2}\right)\zeta(s)$$

It follows that $\zeta(z)$ extends meromorphically to $\mathbb{C}$ with a single simple pole at $z=1$ of residue 1.

**Def 18.6**: The Riemann $\xi$-function is defined by

$$\xi(z) = \frac{1}{2}z(z-1)\pi^{-z/2}\Gamma\left(\frac{z}{2}\right)\zeta(z).$$

**Corollary 18.7**: The $\xi$ function satisfies the cleaner recurrence

$$\xi(s) = \xi(1-s)$$

and extends holo~~mero~~morphically to $\mathbb{C}$ ~~at poles~~.

**Proof**: Recall (Legendre)

$$\sqrt{2\pi}\,\Gamma(2s) = 2^{2s-1}\Gamma(s)\Gamma\left(s+\tfrac{1}{2}\right)$$

$$\Gamma(s)\Gamma(1-s) = \frac{\pi}{\sin \pi s}.$$

The functional equation says

$$\zeta(1-s) = 2(2\pi)^{-s}\underbrace{\Gamma(s)}\cos\left(\frac{\pi s}{2}\right)\zeta(s)$$

$$= \quad = \frac{1}{\sqrt{\pi}}2^{s-1}\Gamma\left(\frac{s}{2}\right)\Gamma\left(\frac{s+1}{2}\right) \quad \text{by Legendre w/ } \frac{s}{2}$$

$$= 2^{1-s}\pi^{-s-1/2}2^{s-1}\Gamma\left(\frac{s}{2}\right)\Gamma\left(\frac{s+1}{2}\right)\underbrace{\cos\left(\frac{\pi s}{2}\right)}\zeta(s)$$

$$\Gamma\left(\frac{s+1}{2}\right)\Gamma\left(\frac{1-s}{2}\right) = \frac{\pi}{\cos \pi s} \qquad 2^{nd} \text{ relation w/ } s = \frac{s+1}{2}$$

$$= -\pi^{-s+\frac{1}{2}}\Gamma(s/2)\frac{\zeta(s)}{\Gamma\left(\frac{1-s}{2}\right)}$$

$$\pi^{-\left(\frac{1-s}{2}\right)}\Gamma\left(\frac{1-s}{2}\right)\zeta(1-s) = \underbrace{-\pi^{-\frac{s}{2}}\Gamma\left(\frac{s}{2}\right)\zeta(s)}$$

$$\tfrac{1}{2}s(s-1) \qquad\qquad \tfrac{1}{2}s(s-1)$$

$$\xi(1-s) \qquad\qquad = \xi(s).$$

For zeros note $\Gamma(\frac{s}{2})$ has poles at $0, -2, -4, \ldots$
but $\zeta(s)$ has zeros at $-2, -4$

$$\zeta(-2) = \zeta(1-3) = \underbrace{\cos\left(\frac{3\pi}{2}\right)}_{=0}\underbrace{\pi^{--}\zeta(3)\Gamma(s)}_{>0}.$$

and $\zeta(s)$ has pole at $-1$, so

$$\Gamma\left(\frac{s}{2}\right)\zeta(s) \quad \text{poles at } 0, 1$$

$$s(s-1)\Gamma\left(\frac{s}{2}\right)\zeta(s) \quad \text{entire} \qquad \square$$

## 18.iii) Proof of the functional equation

**Proof**: Setting $t = nu$ in $\Gamma(u) = \int_0^\infty e^{-u}u^{s-1}\,du$

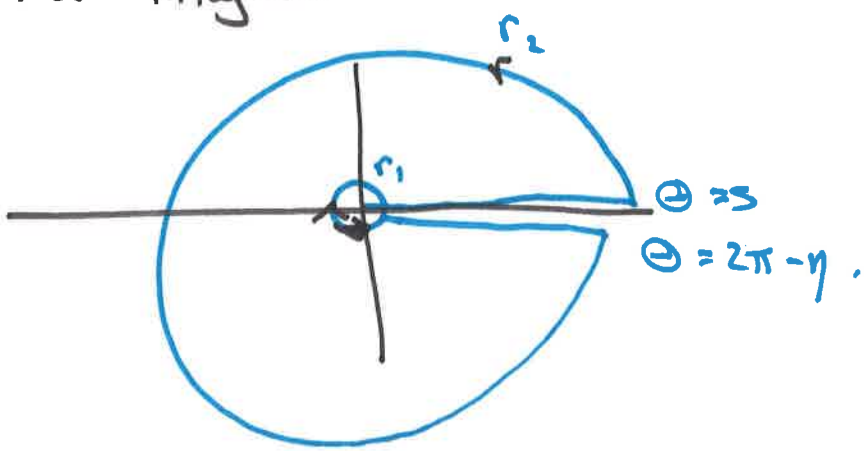
$$n^{-s}\Gamma(s) = \int_0^\infty e^{-nt}t^{s-1}\,dt$$

$$\Gamma(s)\zeta(s) = \sum_{n=1}^{\infty}\int_0^\infty e^{-nt}t^{s-1}\,dt = \int_0^\infty \frac{e^{-t}}{1-e^{-t}}t^{s-1}\,dt$$

Rem: the limit may be exchanged for $\operatorname{Re} s > 1$ by dominated convergence, skipping details, $\boxed{8.4}$

$$= \int_0^\infty \frac{t^{s-1}}{e^t - 1}\, dt.$$

Step 2: Contour Integration




As $\eta \to 0$, multivaluedness of log means $\rightleftarrows$ differ by $e^{s \log t}$

$$\oint_\Gamma \frac{z^{s-1}}{e^z - 1}\, dz = \int_{r_1}^{r_2} \frac{t^{s-1}}{e^t-1}\, dt - e^{2\pi i s}\int_{r_1}^{r_2} \frac{t^{s-1}}{e^t - 1}\, dt$$

$$+ \int_0^{2\pi} \frac{(r_2 e^{i\theta})^{s-1}}{e^{r_2 e^{i\theta}} - 1}\, d\theta - \underbrace{\int [\text{ same w/ } r_1 ]}_{\to 0}.$$

Assume $r_1, r_2 \notin 2\pi \mathbb{N}$. For $0 < r_1, r_2 < 2\pi$, no residues

Claim: $I(s) = (e^{2\pi i s} - 1)\Gamma(s)\zeta(s)$ for

$$I(s) = (e^{2\pi i s} - 1)\int_{r_2}^\infty \frac{t^{s-1}}{e^t - 1}\, dt + \int_{r = r_2} \underline{\qquad}\, dt$$

Proof:

$$I(s) = \underbrace{(e^{2\pi i s} - 1)\int_0^\infty \frac{t^{s-1}}{e^t - 1}\, dt}_{= (e^{2\pi i s} - 1)\Gamma(s)\zeta(s).} + \underbrace{(1 - e^{2\pi i s})\int_0^{r_2} \frac{t^{s-1}}{e^t - 1} + \int_{r = r_2}}_{= \text{res} = 0.}$$

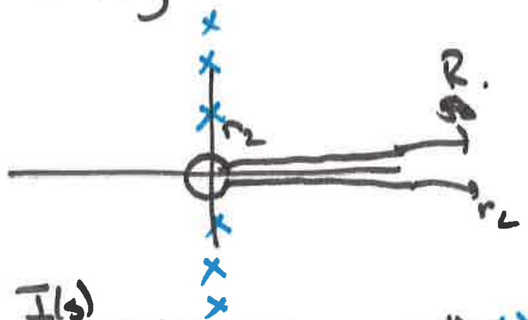
$\Rightarrow I(s)$ is entire b/c $\int_{r_2}^\infty \frac{t^{s-1}}{e^t - 1} < \infty \ \forall s + \int_{\text{compact}}$.

Lecture 19 Analytic Continuation III: Zeros of $\zeta$ and the Riemann hypothesis

## 19.i) Finishing the proof of functional equation

Recall $$I(s) = -(1-e^{2\pi i s})\int_{r_2}^{\infty} \frac{t^{s-1}}{e^t-1}\,dt + \int_{r=r_2} \frac{t^{s-1}}{e^t-1}\,dt$$ ← Ind of $r_2$ for $0 < r_2 < 2\pi$

$$= (e^{2\pi i s}-1)\Gamma(s)\zeta(s).$$



**Def 19.1:** $$\zeta(s) = \frac{I(s)}{(e^{2\pi i s}-1)\Gamma(s)} \Rightarrow \zeta(s)\ \frac{I(s)}{\text{zeros } + +k} \quad k \in 0,1,2,\dots$$

zeros at $s=k$, $k\in\mathbb{Z}$   poles at $s=-k$, $k\in\mathbb{N}$

but $\zeta(s)$ holomorphic so $s=1$ only possible pole, and only simple.

$$\operatorname{Res}_{s=1} = \frac{I(1)}{\Gamma(1)}\operatorname{res}\left(\frac{1}{e^{2\pi i s}-1}\right) = 1.$$

**Step 3:** $$I(s) = -(2\pi)^s\zeta(1-s)\left(e^{\frac{i\pi s}{2}} - e^{\frac{3\pi i s}{2}}\right).$$

**Proof:** take $R\to\infty$. $r_2\to 0$.

$$-I(s) = -(e^{2\pi i s}-1)\int_{r_2}^{\infty}\frac{t^{s-1}}{e^t-1} + \cancel{\int_{r=r_2}} \to 0. + \int_{r=R}\frac{t^{s-1}}{e^{R}-1}\,dt \to 0.$$

$$= -\oint_{C_{R,r_2}}\frac{t^{s-1}}{e^t-1}$$

$$= \operatorname{res}\frac{t^{s-1}}{e^t-1}\Big|_{t=2\pi i k} + \operatorname{res}\frac{t^{s-1}}{e^t-1}\Big|_{t=-2\pi i k}.$$

$$= (2\pi k)^{s-1}e^{\pi i(s-1)/2} + i(2\pi k)^{s-1}e^{\frac{3\pi i s}{2}}$$

$$= -i(2\pi k)^{s-1}e^{\frac{\pi i s}{2}} + i(2\pi k)^{s-1}e^{\frac{3\pi i s}{2}}$$

$\cancel{2\pi i \sum \text{Res}}$

$$2\pi i \sum \text{Res} = \sum_{k=1}^{n} (2\pi)^s k^{s-1} e^{\frac{\pi i s}{2}} - e^{3\frac{\pi i s}{2}}$$

$$\Rightarrow -I(s) = (2\pi)^s \zeta(1-s) e^{\frac{i\pi s}{2}} - e^{3\frac{\pi i s}{2}}$$

$$\Rightarrow -(e^{2\pi i s}-1)\Gamma(s)\zeta(s) = (2\pi)^s \zeta(1-s)\left(e^{\frac{i\pi s}{2}} - e^{\frac{3\pi i s}{2}}\right)$$

$\Rightarrow$ some double angle stuff

$$2(2\pi)^{-s}\Gamma(s)\zeta(s) = \frac{\zeta(1-s)}{\cos\left(\frac{\pi s}{2}\right)} \qquad \square$$

## 19.ii) Euler's Product Formula

**Thm 19.2 (Euler)**

$$\zeta(s) = \prod_{p \text{ prime}} \frac{1}{1-p^{-s}}$$

**Proof sketch**:

$$\zeta(s) = 1 + \frac{1}{2^s} + \frac{1}{3^s} + \dots$$

$$\frac{1}{2^s}\zeta(s) = \frac{1}{2^s} + \frac{1}{4^s} + \frac{1}{6^s}$$

$$\left(1 - \frac{1}{2^s}\right)\zeta(s) = 1 + \frac{1}{3^s} + \frac{1}{5^s} + \frac{1}{7^s} \quad \text{(no factors of 2)}$$

$$\frac{1}{3^s}\left(\qquad\right)\zeta(s) = \frac{1}{3^s} + \frac{1}{9^s} + \frac{1}{15^s} + \dots$$

$$\left(1 - \frac{1}{3^s}\right)\left(1 - \frac{1}{2^s}\right)\zeta(s) = 1 + \frac{1}{5^s} + \frac{1}{7^s} \quad \text{(no factors of 3)}$$

$$\prod_{p \text{ prime}}\left(1 - \frac{1}{p^s}\right)\zeta(s) = 1 \qquad \square$$

**Rem 19.3 : Asymptotic Probabilities**

For $n \in \mathbb{N}$ random, large $\mathbb{P}(p \mid n) = \frac{1}{p}$

$n_1, \ldots n_s$ large random $\mathbb{P}(p \mid n_1, \ldots n_s) = \frac{1}{p^s}$

$p \nmid n_1 \ldots n_s = 1 - p^{-s}$

$\mathbb{P}(n_1 \ldots n_s \text{ pairwise coprime}) = \prod_{p \leq n_s} (1 - \frac{1}{p^s}) \to \frac{1}{\zeta(s)}$

## 19.iii) Zeros and Primes

**Thm 19.3** Let $\pi(x) = \{\# p \text{ prime } p \leq x\}$.
$\exists A, B > 0$ such that

$$A \frac{x}{\log x} \leq \pi(x) \leq B \frac{x}{\log x}.$$

Proof relies on zeros of the $\zeta$-function.

**Thm 19.4** : the only zeros of $\zeta(s)$ outside $0 \leq \mathrm{Re}(s) \leq 1$ are $-2, -4, -6, \ldots$

**Proof** :
$$\pi^{-s/2} \Gamma(s/2) \zeta(s) = \pi^{-(1-s)/2} \Gamma((1-s)/2) \zeta(1-s)$$

$$\zeta(s) = \pi^{s-1/2} \frac{\Gamma(\frac{1-s}{2})}{\Gamma(\frac{s}{2})} \zeta(1-s)$$

no zeros

poles at $-2, -4, \ldots$

no zeros.

For $\mathrm{Re}\, s < 0$,
For $\mathrm{Re}(s) > 1$ known $\neq 0$. $\square$

Conjecture (Riemann) For every $s$ with $\zeta(s) = 0$ and $0 \leq Re(s) \leq 1$ then $Re(s) = \frac{1}{2}$.

Thm (Van Koch '1901) 19.5 :

$$\left| \pi(x) - \int_0^x \frac{1}{\ln t} dt \right| \leq C\, x^{\overset{\text{"}\beta}{|\max Re(s)|}} \log x$$

Corollary 19.6 (Schoenfeld '76) If the Riemann hypothesis holds

$$\left| \pi(x) - \int_0^x \frac{1}{\ln t} dt \right| \leq \frac{1}{8\pi} \sqrt{x} \log x .$$

Thm 19.7 : $\frac{1}{2} \leq \beta \leq 1$.

Thm 19.8 (Hardy - Littlewood) "primes 3 mod 4 are more plentiful than 1 mod 4"

Thm 19.9 (Miller '76) (strong) Riemann hypothesis implies $\exists$ an algorithm to check if $n$ is prime in polynomial time

Rem : later shown regardless of R.H.

# Lecture 20 Conformal Mappings I: definitions and examples

Local vs Global properties of holomorphic functions:

| Local | Global |
|---|---|
| - power series expansions | - Liouville's theorem |
| - removable singularities | - Conformal or biholomorphic equivalence |
| - residues | |

**Question 20.1**: Given $\Omega_1, \Omega_2 \subseteq \mathbb{C}$, when does there exist a holomorphic bijection $f: \Omega_1 \to \Omega_2$ (w/ holomorphic inverse).

**Ex 20.2**: Holomorphic maps are continuous, so $\Omega_1 \simeq \Omega_2$ must be homeomorphic, eg

$\Omega_1 \not\simeq \Omega_2$

**Ex 20.3**: Cannot have $D \simeq \mathbb{C}$, by Liouville.

**Def 20.4**: $\Omega_1, \Omega_2$ are said to be biholomorphic if $\exists f: \Omega_1 \to \Omega_2$ a bijective holomorphic map.

**Lemma 20.5**: If $f: \Omega_1 \to \Omega_2$ is holomorphic and injective, then $f'(z) \neq 0$, and $f^{-1}: \text{Im}(f) \to \Omega_1$ is also holomorphic.

**Proof**: Suppose $f'(z_0) = 0$. Then by Taylor's theorem,

$$f(z) - f(z_0) = a(z - z_0)^k + G(z)$$

for some $k \geq 2$, and $|G(z)| \leq C|z - z_0|^{k+1}$. Write

$$f(z) - f(z_0) - w = F(z) + G(z)$$

where $F(z) = a(z - z_0)^k - w$. For $\delta$, $|w|$ sufficiently small,

$$|F(z)| > |G(z)| \quad \text{on} \quad B_\delta(z_0)$$


$\Rightarrow$ # zeros of $f(z) - f(z_0) - w$

$=$ # zeros of $F(z) \geq 2$.

Thus there are two points $z$ st $f(z) = w+f(z_0)$. If this were a double root, would have $f'(z)=0$ (but zeros of $f'$ are isolated, so reducing $\delta$ eliminates this). $\rightarrow\leftarrow$ to injective.

Cauchy Riemann says $df = \begin{pmatrix} a & -b \\ b & a \end{pmatrix}$ as a real map

$df^{-1} = \begin{pmatrix} a & b \\ -b & a \end{pmatrix}$ also satisfies Cauchy-Riemann $\square$.

## 20.ii) Conformal Mappings

**Def 20.6**: A map $f: \underset{\mathbb{R}^2}{\underset{\cap}{U}} \to \underset{\mathbb{R}^2}{\underset{\cap}{V}}$ is said to be conformal if it preserves angles.

$\iff \langle u,v\rangle = 0 \implies \langle df_z u, df_z v\rangle = 0 \quad \forall\, u,v \in \mathbb{R}^2$

$\iff df_z^T df_z = e^{u(z)}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ for a function $u(z): U \to \mathbb{R}$.

**Prop 20.7**: A biholomorphic map is conformal $f: \Omega_1 \to \Omega_2$ (if and only if)

**Proof**: $\Rightarrow$ By Cauchy-Riemann $df_z = \begin{pmatrix} a & -b \\ b & a \end{pmatrix}$

so $df_z^T df_z = \begin{pmatrix} a & b \\ -b & a \end{pmatrix}\begin{pmatrix} a & -b \\ b & a \end{pmatrix} = |f'(z)|^2 \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ $\square$

$\Leftarrow$ any orthogonal $df$ is of this form.

**Rem 20.8**: This implication is only true in $\dim_{\mathbb{R}} = 2$, $\dim_{\mathbb{C}} = 1$,

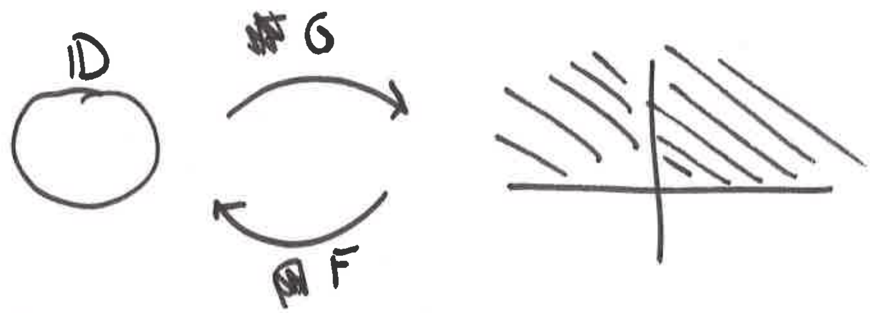
in higher dimensions ~~biholomorphic~~ both ~~is~~ are a <u>much</u> stronger condition, and there is no inclusion $\subseteq$ or $\supseteq$.

## 20.iii) Examples

Let $\mathbb{H} = \{\mathrm{Im}(z) > 0\}$ be the upper half plane.

$\mathbb{D}$ = unit disk

**Prop 20.9**




By $F = \frac{i-z}{i+z}$ and $G(w) = i\frac{1-w}{1+w}$ is a biholomorphism.

**Proof**: If $z = (a+ib)$ w/ $b>0$

$$|F(z)| = \left|\frac{i-a-ib}{i+a+ib}\right| = \frac{|a+(b-1)i|}{|a+(b+1)i|} < 1 \quad \text{so lands in } \mathbb{D}$$

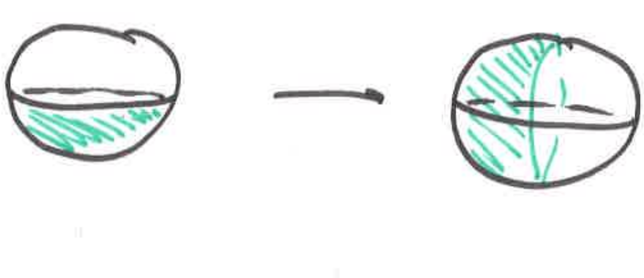
$$\mathrm{Im}\, G(w) = \mathrm{Re}\left[\frac{1-x-iy}{1+x+iy}\right] = \frac{1-u^2-v^2}{1+u^2+v^2} > 0 \quad \text{so land in } \mathbb{H}.$$

$$F\circ G(w) = \frac{i - i\left(\frac{1-w}{1+w}\right)}{i + i\left(\frac{1-w}{1+w}\right)} = \frac{1+w-1+w}{1+w+1-w} = \frac{2w}{2} = w.$$

$$G\circ F(z) = z.$$

**Rem 20.10**: The Riemann Sphere or $\mathbb{CP}^1$ is $\mathbb{C}\cup\{\infty\}$


Then this map is just



**Rem 20.11** : from the picture it is obvious

$$F(\partial \mathbb{H}) = \partial \mathbb{D} \quad \text{and vice-versa.}$$

and this is easy to check with the formulas

More generally

$$z \longmapsto \frac{az+b}{cz+d}$$

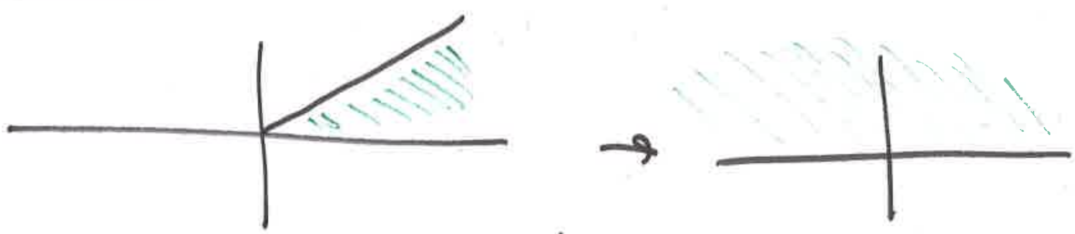
fractional linear transformations are all conformal.

**Ex 20.12** : $z \longmapsto z + c$

$z \longmapsto cz$ are conformal / biholomorphisms of $\mathbb{C}$

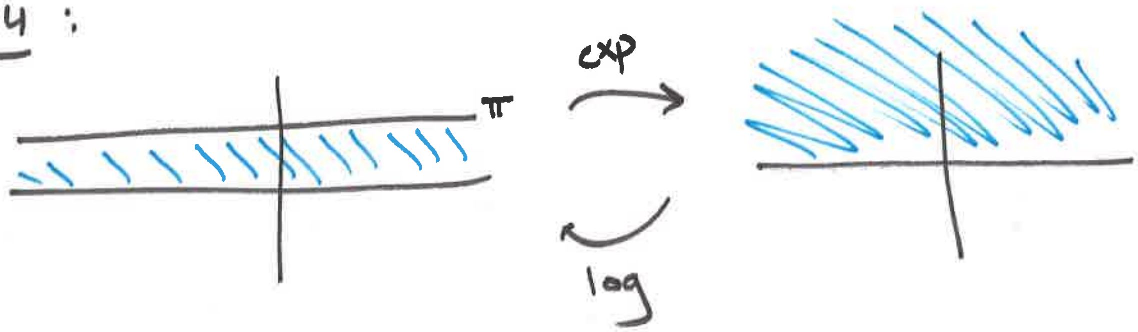
$c \in \mathbb{C}$.

**Ex 20.13** : $z \longmapsto z^n$

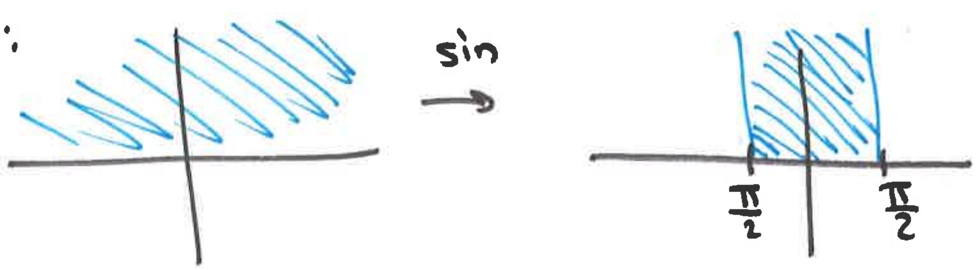


is conformal w/ $z^{1/n}$ (principal branch) the inverse.

**Ex 20.14** :




**Ex 20.15** :



# Lecture 21 Conformal Mappings II: Schwartz Lemma

<u>Question 21.1</u>: Can we completely describe the space

$$\mathrm{Aut}(\mathbb{D}) = \{ f : \mathbb{D} \to \mathbb{D} \text{ biholomorphic} \}?$$

## 21.i) Automorphisms of the disk

(Schwartz Lemma)
<u>Lemma 21.2</u>: Let $f \in \mathrm{Aut}(\mathbb{D})$ with $f(0)=0$. Then

i) $|f(z)| \leq |z| \quad \forall z$ (contraction) ii) $|f'(0)| \leq 1$

ii) If equality $|f(z_0)| = |z_0|$ holds for some $z_0$ in $\mathring{\mathbb{D}}$.
OR if $|f'(0)|$ then $f = e^{i\theta}$ is a rotation.

<u>Proof</u>: Since $f(z)$ is holomorphic, $f(0)=0$,

$$f(z) = a_1 z_1 + a_2 z^2 + \ldots$$

so $\frac{f(z)}{z}$ is also holomorphic. Thus since $|f(z)| \leq 1$ (lands in disk)

$$\frac{|f(z)|}{|z|} \leq \frac{1}{r}$$

for $|z|=r$. And by max principle on $D_r(0)$, true for $|z| \leq r$.
Letting $r \to 1$ shows i).

ii) $$|f'(0)| = \left| \lim_{z \to 0} \frac{f(z) - f(0)}{z} \right| = \left| \lim_{z \to 0} \frac{f(z)}{z} \right| \leq 1$$

iii) If $f'(0) = 1$ then $f'(z)$ attains an interior max, so is const

$$\Rightarrow f = c z \quad w/ \ |c| = 1.$$

Same if $|f(z)| = |z|$ for $z_0$ in $\mathring{\mathbb{D}}$. $\square$

<u>Ex 21.3</u>: $z \longmapsto e^{i\theta} z$ is an automorphism (obviously)

<u>Ex 21.4</u>: Consider $\psi_\alpha(z) = \frac{\alpha - z}{1 - \bar{\alpha} z}$ for $|\alpha| < 1$.

This is holomorphic and we claim $\psi_\alpha : \mathbb{D} \to \mathbb{D}$. Note if $|z|=1$

$$\psi_\alpha(e^{i\theta}) = \frac{\alpha - e^{i\theta}}{e^{i\theta}(e^{-i\theta} - \bar{\alpha})} = e^{i\theta} \frac{w}{\bar{w}} \qquad w = \alpha - e^{i\theta}$$

$\Rightarrow |\psi_\alpha(e^{i\theta})| = 1$, so $\psi_\alpha : \partial\mathbb{D} \to \partial\mathbb{D}$ by max principle $\mathbb{D} \to \mathbb{D}$.

Moreover

$$\psi_\alpha \circ \psi_\alpha(z) = \frac{\alpha - \dfrac{\alpha - z}{1-\bar\alpha z}}{1 - \bar\alpha \dfrac{\alpha - z}{1 - \bar\alpha z}}$$

$$= \frac{\alpha - |\alpha|^2 z - \alpha + z}{1 - \bar\alpha z - |\alpha|^2 + \bar\alpha z} = \frac{(1-|\alpha|^2)z}{1-|\alpha|^2} = z = \mathrm{Id}.$$

And $\quad \psi_\alpha(0) = \alpha$.

<u>Thm 21.5</u> : If $f \in \mathrm{Aut}(\mathbb{D})$ then $\exists \theta \in [0, 2\pi)$, $\alpha \in \mathbb{D}$ st

$$f(z) = e^{i\theta} \frac{\alpha - z}{1 - \bar\alpha z}$$

ie $\quad \mathrm{Aut}(\mathbb{D}) = S^1 \times \mathbb{D}$.

<u>Proof</u> : $\exists!\ \alpha \in \mathbb{D}$ such that $\alpha = 0$, so by replacing $f \to f \circ \psi_\alpha$ it suffices to consider $f(0) = 0$. hence $f^{-1}(0) = 0$ also.

$$|z| = |f^{-1} f(z)|$$

by Schwartz

$$\leq |f(z)| \leq |z|$$

so must have $|f(z)| = z$ for all $z \Rightarrow f$ is a rotation..
Thus original $f$ is $e^{i\theta}\psi_\alpha$. $\square$

<u>Corollary 21.6</u> : If $f \in \mathrm{Aut}(\mathbb{D})$ and $f(0) = 0$ then $f$ is a rotation.

<u>Rem 21.7</u> : Note that all $f \in \mathrm{Aut}(\mathbb{D})$ are of the form

$$f(z) = \frac{az + b}{cz + d}$$

ie are moblus transformations.

<u>Rem 21.8</u> : The action $\mathrm{Aut}(\mathbb{D}) \to \mathbb{D}$ is <u>transitive</u> in the sense that $\forall \alpha, \beta \in \mathbb{D}\ \exists f \in \mathrm{Aut}$ w/ $f(\alpha) = \beta$. Take

$$f = \psi_\beta \psi_\alpha^{-1}.$$

## 21.ii) Automorphisms of $\mathbb{H}$

Recall that $F = \frac{i-z}{i+z}$ is a biholomorphism $\mathbb{H} \xrightarrow{F} \mathbb{D}$.

There is therefore a map

$$\Gamma : \mathrm{Aut}(\mathbb{D}) \longrightarrow \mathrm{Aut}(\mathbb{H})$$
$$\varphi \longmapsto F^{-1} \circ \varphi \circ F.$$

**Lemma 21.9** : $\Gamma$ defines a (group) isomorphism.

**Proof** : First note that

$$\Gamma(\varphi_1 \circ \varphi_2) = F^{-1} \varphi_1 \varphi_2 F$$
$$= (F^{-1} \varphi_1 F)(F^{-1} \varphi_2 F) = \Gamma(\varphi_1)\Gamma(\varphi_2)$$

so $\Gamma$ is a homomorphism. Next, note

$$\xi \longmapsto F \circ \xi \circ F^{-1} \quad \text{for } \xi \in \mathrm{Aut}(\mathbb{H})$$

is clearly the inverse.

**Prop 21.10** : $\mathrm{Aut}(\mathbb{H}) = \left\{ \frac{az+b}{cz+d} \;\middle|\; \det\begin{pmatrix} a & b \\ c & d \end{pmatrix} = 1, \; a,b,c,d \in \mathbb{R} \right\}$.

**Proof** : **Step 1** : If $f$ has the above form,

$$\mathrm{Im}(f(z)) = \frac{(ad-bc)\,\mathrm{Im}(z)}{|cz+d|^2} = \frac{\mathrm{Im}(z)}{|cz+d|^2} > 0,$$

so maps $\mathbb{H}$ to $\mathbb{H}$.

and $f_1 \circ f_2$ is just matrix multiplication, so each $f$ has inverse w/ inverse matrix ($\det = 1$).

**Step 2** : The action is transitive, ie $\exists f_y : \mathbb{H} \to \mathbb{H}$ such that $f_y(y) = i$. (so $\forall y_1, y_2 \; \exists \; f_{y_2}^{-1} f_{y_1}$ taking $y_1$ to $y_2$)

One has $\mathrm{Im}(f_y(y)) = \dfrac{\mathrm{Im}(y)}{|cy|^2}$ w/ $d=0$, so choose

$c$ st $|cy| = 1$, then $f_y(y) = \frac{b}{a} + i$.

It is easy to check

$$m_1 = \begin{pmatrix} 0 & -c^{-1} \\ c & 0 \end{pmatrix}$$

implements this, and

$$m_2 = \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix}$$

is a translation of the real part by b. So

$$f_y = m_2 m_1$$

<u>Step 3</u>: If $\theta \in \mathbb{R}$ note that

$$m_\theta = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$$

is in the group, and $\Gamma(m_\theta) = e^{-2i\theta} : \mathbb{D} \to \mathbb{D}$.

<u>Step 4</u>: It suffices to assume $f(i) = i$. Then

$$\Gamma(f)(0) = 0$$

so is a rotation. $\Rightarrow m = m_\theta m_1 m_2$ for some $m_0 m_1 m_2$.

## 21.iii) : Lie subgroups :

A <u>Lie group</u> (compact) is a (quotient of) matrix subgroup of $GL(n,\mathbb{C})$ (not actually, but for our purposes this is an okay def)

The groups of moblus transformations

$$PSL(2,\mathbb{C}) = \left\{ \frac{az+b}{cz+d} \;\middle|\; \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in GL(2,\mathbb{C}) \right\} \Big/ \{\pm 1\}.$$

give same moblus trans.

has two subgroups:

$$PSL(2,\mathbb{R}) = \mathrm{Aut}(\mathbb{H}) = \left\{ \frac{az+b}{cz+d} \;\middle|\; \begin{pmatrix} a, & b \\ c & d \end{pmatrix} \in GL(2,\mathbb{R}) \ \det = 1 \right\} \Big/ \{\pm 1\}$$

$$PSU(1,1) = \mathrm{Aut}(\mathbb{D}) = \left\{ \frac{az+b}{cz+d} \;\middle|\; A^{-1}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}A = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \text{ and } \det A = 1 \right\} \Big/ \{\pm 1\}$$

And so we have a diagram

$$\begin{tikzcd}
PSL(2,\mathbb{R}) = \mathrm{Aut}(\mathbb{H}) \arrow[d, shift right, "F"'] \arrow[dr, hook] & \\
PSU(1,1) = \mathrm{Aut}(\mathbb{D}) \arrow[u, shift right] \arrow[r, hook] & PSL(2,\mathbb{C}) = \mathrm{Aut}(\mathbb{CP}^1)
\end{tikzcd}$$

of conjugate subgroups.

# Lecture 22 The Riemann Mapping Theorem

Recall that if $\Omega \simeq \mathbb{D}$ are biholomorphic then $\Omega$ must be topologically a disk, ie $\Omega$ is connected and simply-connected.

Also, $\Omega \neq \mathbb{C}$ by Liouville. $\Omega \subsetneq \mathbb{C}$ is a proper subset in this case.

**Thm 22.1** (Riemann Mapping Theorem) Suppose $\Omega \subsetneq \mathbb{C}$ is a proper, connected, simply-connected open subset. $\forall z_0 \in \Omega$, $\exists !$ biholomorphism

$$\Omega \overset{F}{\simeq} \mathbb{D}$$

s.t. $F(z_0) = 0$ and $F'(z_0) > 0$.

**Corollary 22.2**: Any $\Omega_1, \Omega_2$ satisfying the assumptions are biholomorphic.

**Proof**: $\Omega_1 \overset{F_1}{\simeq} \mathbb{D} \overset{G^{-1}}{\simeq} \Omega_2$.

**Proof** (of uniqueness) If $F, G$ both satisfy the conclusions,

$$F \circ G^{-1} : \mathbb{D} \to \mathbb{D}$$

is an automorphism fixing $0$, so a rotation by Schwartz. Since $F'(z_0) = e^{i\theta}$ must have $\theta = 0$. $\square$

**Rem 22.3**: This is extremely false in higher dimensions

$$\left\{\begin{matrix}\text{Homotopy}\\ \text{equivalence}\end{matrix}\right\} \subsetneq \left\{\text{Homeomorphic}\right\} \subsetneq \left\{\text{smooth}\right\} \subsetneq \left\{\text{conformal}\right\}$$

$$\{\text{conformal}\} = \{\text{biholomorphic}\}$$

$$\left\{\begin{matrix}\text{Homotopy}\\ \text{equivalence}\end{matrix}\right\} \xrightarrow{\text{Riemann mapping}} \{\text{biholomorphic}\}$$

In fact, on $\mathbb{C}^n$ for $n \geq 2$, the box is not biholomorphic with the disk.

**Thm 22.4** (Liouville Rigidity): If $\Omega_1 \overset{F}{\cong} \Omega_2 \subseteq \mathbb{R}^n$ for $n>2$, are conformally equivalent, then $F$ is a composition of reflections, translations, orthogonal, and special conformal transformations (conformal groups is finite-dimensional!)

## 22.ii) Proof of the Theorem

**Idea of Proof**:
- Choose $f: \Omega \to \mathbb{D}$ holomorphic and injective (show these exist)
- Order such maps by inclusion of image
- extract holomorphic "supremum".

**Def 22.5**: A family of functions $\mathcal{F} \subseteq hol(\Omega; \mathbb{C})$ is said to be locally pre-compact if $\forall \{f_n\} \subseteq \mathcal{F}$ and $K \subset\subset \Omega$ compact, a subsequence

limit need not be in $\mathcal{F}$

$$f_{n_k} \to f$$

converges uniformly on $K$.

Recall (or learn) the following theorem:

**Thm 22.6** (Arzela-Ascoli)
Suppose a family of functions $\mathcal{F}$ is
1) uniformly bounded ($\forall K \subseteq \Omega, \exists B$ st $|f(z)| \leq B \ \forall f \in \mathcal{F}$)
2) equicontinuous ($\forall \varepsilon > 0, \exists \delta$ st $|z-w| < \delta \Rightarrow |f(z) - f(w)| < \varepsilon \ \forall f \in \mathcal{F}$.)

then $\mathcal{F}$ is sequentially compact (uniform limits)

**Proof**: Diagonalization argument.

**Corollary 22.7**: If $\mathcal{F}$ consists of holomorphic functions and is uniformly bounded and equicontinuous, the $\mathcal{F}$ is locally pre-compact.

**Proof**: uniform limit of holomorphic functions is holomorphic. $\boxed{22.3}$

**Thm 22.8 (Montel)**: If $\mathcal{F}$ is uniformly bounded and holomorphic, it is automatically equicontinuous.

**Proof**: If $z, w \in K$, and $r$ st $B_r(z) \subset \Omega \ \forall z \in K$,

$$|f(z)-f(w)| = \left|\frac{1}{2\pi i}\oint_{\partial D} f(\zeta)\left[\frac{1}{\zeta - z} - \frac{1}{\zeta - w}\right] d\zeta\right|$$

$$\leq c \left|\oint_{\partial D} f(\zeta) \underbrace{\left[\frac{|z-w|}{|\zeta - z||\zeta - w|}\right]}_{\leq \frac{|z-w|}{r^2}} d\zeta\right|$$

$$\leq \frac{1}{2\pi}\frac{2\pi r}{r^2} B|z-w|$$

so take $\delta < \frac{\varepsilon r}{B}$. □

22.iii)

**Proof of Riemann Mapping**

**Lemma 22.9**: Suppose $\Omega \subseteq \mathbb{C}$ is connected and open. If $\{f_n\} : \Omega \to \mathbb{C}$ are holomorphic and injective, and $f_n \to f$ uniformly on compact subsets ($\Rightarrow f$ holomorphic) then $f$ is injective or const.

**Proof**: Suppose $f(z_1) = f(z_2)$. Set $g_n^{(z)} = f_n(z) - f_n(z_1)$

Then
$$g_n \to g = f_n(z) - f_n(z_1) \text{ uniformly on compact subsets.}$$

Then $z_2$ is an isolated zero (else $g \equiv 0$), hence

$$1 = \frac{1}{2\pi i}\oint_\gamma \frac{g'(\zeta)}{g(\zeta)} d\zeta \quad \text{(argument principle)}$$

where $\gamma$ has no zeros of $g$, so $g', \frac{1}{g}$ converge uniformly. But then

by inj. of $g_n$ $\longrightarrow$ $0 = \lim_{n\to\infty} \frac{1}{2\pi i}\oint_\gamma \frac{g_n'}{g_n} d\zeta = \frac{1}{2\pi i}\oint \frac{g'}{g} d\zeta = 1 \rightarrow\leftarrow$ □.

Proof (of Riemann mapping)

Step 1 : $\exists F : \Omega \to F(\Omega) \subseteq \mathbb{D}$ holomorphic and injective.

- Let $\alpha \in \mathbb{C} \setminus \Omega$, and set $f(z) = \log(z-\alpha)$. (since $z-\alpha \neq 0$ can choose a log)
- Note $f$ is injective because $e^{f(z)} = z - \alpha$
- $f(z) \neq f(w) + 2\pi i$, else $e^{f(z)} = e^{f(w)} = w - \alpha = z - \alpha$ so $z = w$.
- In fact $|f(z) - f(w) - 2\pi i| > \varepsilon > 0$, else $\exists$ sequence $z_n \to w$ but this implies $f(z_n) \to f(w)$ $\to\leftarrow$ b/c differs by $2\pi i$.
- Hence $F(z) = \dfrac{1}{f(z) - f(w) - 2\pi i}$ is bounded and injective.

by scaling and translation, can arrange $\hat{F} = \frac{1}{C} F + \alpha$ lands in $\mathbb{D}$.

Step 2 : Replace $\Omega$ by $F(\Omega) \subseteq \mathbb{D}$.

Set
$$\mathcal{F} = \{f : \Omega \to \mathbb{D} \text{ holomorphic, injective, } f(0) = 0\}.$$

- There exist $C_0$ st $|f'(0)| < C_0$ for $f \in \mathcal{F}$ because by Cauchy $|f'(0)| \leq \frac{C}{r} \|f\|_{C^0} \leq \frac{C}{r}$ on circle of radius $r$ since $f \in \mathbb{D}$.
- Set $s = \sup_{f \in \mathcal{F}} |f'(0)|$. $s \geq 1$ since $\mathrm{Id} \in \mathcal{F}$.
- By Montel's theorem, $\mathcal{F}$ is locally pre-compact, since its uniformly bounded by 1. Take $f_n$ st $f_n'(0) \to s$. Then $f_n \to f$ for some $f$, uniformly on compact subsets.
- By Lemma 22.9 $f$ is injective (not const b/c $f'(0) \neq 0$).
- by continuity $f(\Omega) \subseteq \overline{\mathbb{D}}$, max principle prevents $=$ since $\Omega$ is open.
- Therefore $f \in \mathcal{F}$ since $f(0) = 0$ by ~~definition~~ convergence.

<u>Step 3</u> : Claim $f : \Omega \to \mathbb{D}$ is biholomorphic.

- Inj ✓, suppose $\exists \alpha \in \mathbb{D}$ st $\alpha \notin \mathrm{Im}\, f$.
- Let $\psi_\alpha$ be the aut of $\mathbb{D}$ sending $\alpha$ to $0$. Then
$$U = (\psi_\alpha \circ f)(\Omega)$$
is simply-connected and doesn't contain $0$, so $\exists \sqrt{w}$ on $U$.
- Set
$$F = \psi_{\sqrt{\alpha}} \circ \sqrt{\phantom{x}} \circ \psi_\alpha \circ f.$$
- $F \in \mathcal{F}$ b/c its holomorphic, $f(0) = 0 \xrightarrow{\psi_\alpha} \alpha \to \sqrt{\alpha} \xrightarrow{\psi_{\sqrt{\alpha}}} 0$ and all land in $\mathbb{D}$.
- Note
$$f = \psi_\alpha^{-1} \circ (\psi_{\sqrt{\alpha}} \circ F)^2 = \Phi \circ F \quad \text{where } \Phi = \psi_\alpha^{-1} \circ (\;)^2 \circ \psi_{\sqrt{\alpha}}^{-1}$$
- $\Phi(0) = 0$ and $\Phi : \mathbb{D} \to \mathbb{D}$ (but isn't injective), hence by Schwartz lemma, $|\Phi'(0)| < 0$.
- $$f'(0) = \Phi'(0) F'(0) \implies |f'(0)| < |F'(0)|$$
$\to\leftarrow$ maximality of $f'(0)$. since $f \in \mathcal{F}$.

# Lecture 23 The Mittag-Leffler Theorem



one dude

Recall

We saw that $f: \mathbb{C} \to \mathbb{C}$ entire can be constructed with prescribed zeros (Weierstrauss), and actually this determines the function up to $e^{g(z)}$ (Hadamard).

Question 23.1: Given data of poles, can we construct an entire (Mittag-Leffler problem) meromorphic function with those poles.

Def 23.2: If $f$ is meromorphic with a pole at $z_0$,

$$f(z) = \underbrace{\frac{a_{-m}}{(z-z_0)^m} + \frac{a_{-m+1}}{(z-z_0)^{m-1}} + \cdots + \frac{a_{-1}}{(z-z_0)}}_{P_{z_0}(z)} + g(z)$$

$P_{z_0}(z)$ is called the principal part at $z_0$.

Theorem 23.3 (Mittag-Leffler) Suppose $\{z_n\} \subset \mathbb{C}$ is a sequence of distinct points with $|z_n| \to \infty$. Let $P_n$ be polynomials with $P_n(0)=0$. Then there exists an entire meromorphic $f: \mathbb{C} \to \mathbb{C}$ with poles at $z_n$ whose principal parts are

$$P_{z_n}(z) = P_n\left(\frac{1}{z-z_n}\right).$$

Moreover, any such function has the form

$$f(z) = \sum_{n=1}^{\infty} \left[ P_n\left(\frac{1}{z-z_n}\right) + Q_n(z) \right] + g(z)$$

- $Q_n$ polynomials, $g(z)$ entire.

converges uniformly on compact sets.

# 23.ii) Examples of Mittag-Leffler Products

**Ex 23.4**: $\dfrac{\pi^2}{\sin(\pi z)^2} = \displaystyle\sum_{k=-\infty}^{\infty} \frac{1}{(z-k)^2}.$

$\sin(\pi z)$ has simple poles at $\mathbb{Z}$, so consider

$$\sum_{k=-\infty}^{\infty} \frac{1}{(z-k)^2}$$

For fixed $z$, $\frac{1}{(z-k)^2} \leq \frac{4}{k^2}$ once $k \geq 2|z|$, so converges uniformly.

$$\frac{\pi^2}{\sin(\pi z)^2} - \sum_{k=-\infty}^{\infty} \frac{1}{(z-k)^2} = g(z) \text{ with } g \text{ entire}$$

(removable sing at $\mathbb{Z}$).

Thus we claim $g \equiv 0$. Note $g(z) = g(z+1)$ since other two have this property.

Claim $|g(x+iy)| < C$ for $|x| \leq \frac{1}{2}$.

i) $\sin(\pi z) = \frac{1}{2i}\left(e^{i\pi x - \pi y} - e^{-i\pi x + \pi y}\right)$ so $|\ | \to \infty$ for $|y| \to \infty$

$\frac{1}{|\sin(\pi z)|^2} \to 0$ uniformly.

ii) $\left|\frac{1}{(z-k)^2}\right| \leq \frac{4}{y^2+k^2}$ for $|x| \leq \frac{1}{2}$

$$\leq \int_k^{k+1} \frac{4}{x^2+y^2}\,dx$$

$$\left|\sum \frac{1}{(z-k)^2}\right| \leq \frac{1}{|z|^2} + \sum \int_k^{k+1} \frac{4}{x^2+y^2}\,dx$$

$$+ \int_{-\infty}^{\infty} \frac{4}{x^2+y^2}\,dx \to 0 \text{ as } |y| \to \infty.$$

<u>Ex 23.5</u> : $\pi\cot(\pi z) = \frac{1}{z} + \sum_{k=1}^{\infty} \frac{2z}{z^2-k^2}$

$\pi\cot(\pi z)$ poles at $\mathbb{Z}$ w/ residue 1.

$\sum_{k=-\infty}^{\infty} \frac{1}{z-k}$ not summable

this is what $Q_n(z)$ is for!

$\frac{1}{z-k} = -\frac{1}{k}\left[1 + \frac{z}{k} + \ldots\right]$ correct by add terms of power series at 0.

$$\sum_{\substack{k=-\infty \\ k\neq 0}}^{\infty} \frac{1}{z-k} - \left(-\frac{1}{k}\right) = \frac{1}{z} + \sum_{k\neq 0} \frac{z}{k(z-k)} \quad \text{now converges}$$
$$= \frac{1}{z} + \sum_{k\geq 1} \frac{2z}{z^2-k^2}$$

And no entire function (check $\partial_z \pi\cot(\pi z) = \frac{\pi^2}{\sin(\pi z)}$ ).

<u>Rem 23.6</u> : Mittag-Leffler holds also for $\Omega \subseteq \mathbb{C}$ non-compact with $\{z_n\}$ no accumulation points. The general proof is harder and non-constructive.

<u>Proof</u> (of Mittag-Leffler)

By adding finite terms, can assume $|z_n| \geq 1$ for all $n$.

We will choose $Q_n(z)$ such that

$$\left|P_n\left(\frac{1}{z-z_n}\right) + Q_n(z)\right| \leq \frac{1}{2^n} \quad \text{provided } \left|\frac{z}{z_n}\right| \leq \frac{1}{2}.$$

Given this $\sum P_n\left(\frac{1}{z-z_n}\right) + Q_n(z)$ converges absolutely on any compact region, since $\left|\frac{z}{z_n}\right| \leq \frac{1}{2}$ for $n$ sufficiently large.

Write

$$\frac{1}{(z-z_n)^k} = \frac{(-1)^k}{z_n^k}\left[\frac{1}{\left(1-\frac{z}{z_n}\right)^k}\right]$$

$$= \left(\frac{-1}{z_n}\right)^k \left[\frac{1}{1 - k\frac{z}{z_n} + \binom{k}{2}\left(\frac{z}{z_n}\right)^2 + \cdots}\right]$$

$$= \frac{-1}{z_n} \sum_{k=1}^{\infty} b_{kj} \left(\frac{z}{z_n}\right)^j$$

converges for $\left|\frac{z}{z_n}\right| \leq \frac{1}{2}$, so by truncating sum, may assume

$$\left|\frac{1}{|z-z_n|^k} - Q_n\left(\frac{z}{z_n}\right)\right| \leq C_n \left(\frac{z}{z_n}\right)^{j+1}$$

so take $j \geq J(n)$ such that $\frac{C_n}{2^{J(n)+1}} \leq \frac{1}{2}$. ∎

# Lecture 27 Harmonic functions



**Recall**:

$$\left\{\begin{matrix}\text{holomorphic}\\ \underbrace{\bar\partial f = 0}_{\text{1st order}}\end{matrix}\right\} \subseteq \left\{\begin{matrix}\text{Harmonic}\\ \underbrace{\Delta f = 0}_{\text{second order}}\end{matrix}\right\} \subseteq \left\{\begin{matrix}\text{Sol. of}\\ \text{elliptic PDE}\end{matrix}\right\}$$

## 27.i: Basic Definitions

**Def 27.0**: A function $u : \Omega \to \mathbb{R}$ is harmonic if $-\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)u = \Delta u = 0$.

**Lemma 27.1**: If $f = u + iv$ is holomorphic, then $u, v$ are harmonic.

**Proof**:

$$0 = -\partial\bar\partial f = \tfrac{1}{2}\tfrac{1}{2}(\partial_x - i\partial_y)(\partial_x + i\partial_y) f$$

$$= \tfrac{1}{4}(\partial_x^2 - i\partial_y\partial_x + i\partial_x\partial_y + \partial_y^2) f$$

$$= \tfrac{1}{4}\Delta(u + iv) \qquad \square$$

**Lemma 27.2**: Any $u : \Omega \to \mathbb{R}$ harmonic is the real part of a holomorphic function, only any simply connected $\Omega' \subseteq \Omega$.

**Proof**: Set $g(z) = (\partial_x u)(x,y) - i(\partial_y u)(x,y)$

and consider $f = u(z_0) + \int_{\gamma : z_0 \to z} g(z)\,dz$

$\bar\partial g = \bar\partial\partial(u) = \frac{1}{4}\Delta u = 0$, so $f$ is holomorphic. $U = \mathrm{Re}(f)$, $V = \mathrm{Im}(f)$

$$f'(z) = \partial f = \tfrac{1}{2}(\partial_x - i\partial_y)(U + iV) \qquad \begin{matrix}\partial_x U = \partial_y V\\ \partial_y U = -\partial_x V\end{matrix}$$

$$= \partial_x U - \partial_y U$$

but $f'(z) = g(z) = \partial_x u + - i\partial_y u$, and $U = u$ at $z_0$ so

$U = u$. $\square$

## 27.ii : Properties of Harmonic Functions



<u>Thm 27.3</u> : If $\begin{matrix} u \in C^2 \\ u : \Omega \to \mathbb{R} \end{matrix}$ simply connected in harmonic, then it satisfies

i) (Elliptic Regularity) $u$ is smooth and real-analytic

ii) (Mean Value property) For a disk of radius $r$ in $\Omega$,

$$u(x) = \frac{1}{2\pi r} \oint_{\partial D} u(\theta)\, d\theta.$$

iii) (Maximum principle) If $u$ attains a maximum in the interior, it is constant.

iv) (Liouville) If $f : \mathbb{R}^2 \to \mathbb{R}$ is harmonic and bounded, it is constant.

v) (Removable Singularity) If $u\!\!\!/ f : \Omega \setminus \{x_0\} \to \mathbb{R}$ is harmonic and bounded, then $\exists$ an extension $\bar{u} : \Omega \to \mathbb{R}$ harmonic.

<u>Proof</u> : $u = \mathrm{Re}(f)$ $f$ holomorphic, and these are all known properties of holomorphic functions.

<u>Rem 27.4</u> : All of these can be proved using only the $2^{nd}$ order equation $\Delta u = 0$. In particular, they hold in odd dimensions where $\bar{\partial}$ doesn't make sense.

## 27.iii) Applications of Harmonic Functions

<u>Ex 27.5</u> (Electrostatics) : Suppose $(\Omega, \partial\Omega)$ is a region in space w/ voltage $V_0(x)$ fixed on $\partial\Omega$. Then the electric field potential (voltage) satisfies

$$\begin{cases} \Delta V = 0 & \text{on } \Omega \\ V|_{\partial\Omega} = V_0(x) & \text{on } \partial\Omega. \end{cases}$$

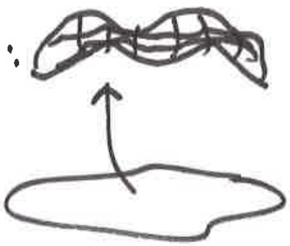
<u>Ex 27.6</u> (Diffusion Eq) : Let $c$ denote the density of a material diffusing in ~~salt~~ water in $(\Omega, \partial\Omega)$, with material source $f$. Then

$$\begin{cases} -\Delta c = f & \text{on } \Omega \\ c|_{\partial\Omega} = 0 & \text{or } \partial_\nu c|_{\partial\Omega} = 0. \end{cases}$$

**Ex 27.7** (Heat Eq) : Heat density $u_0(x)$ on $\Omega$ with constant temp $T_0$ at boundary, heat evolution satisfies

$$\begin{cases} \partial_t u + \Delta u = 0 & \text{on } \Omega \\ u|_{\partial\Omega} = T_0 \end{cases}$$

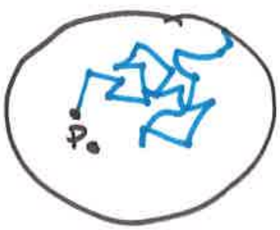
**Ex 27.8** (Minimal Surfaces) :



An embedded surface minimizes area for its boundary when $\Delta u_i = 0$

$(u_1, u_2, u_3) : \mathbb{D} \to \mathbb{R}^3$

**Ex 27.9** (Brownian Motion)




Consider random walks from $p_0$ by the algorithm

while $|p| < 1$,

$p := p + u$

$u \sim$  gaussian random variable

end

For $v \in \partial\Omega$, let $\mu_p(v) : \partial\Omega \to \mathbb{R}$ be the probability path exits at $v$.

Then $$u(p) = \int_{\partial\Omega} g(v)\mu_p(v)\, dv$$ satisfies

$$\begin{cases} \Delta u = 0 & \text{on } \Omega \\ u|_{\partial\Omega} = g & \text{on } \partial\Omega. \end{cases}$$

$\square$

**Def 27.10** : The Dirichlet Problem on $\Omega$ is to find a harmonic function $u : \Omega \to \mathbb{R}$ st

$$\begin{cases} \Delta u = 0 & \text{on } \Omega \\ u|_{\partial\Omega} = f & \text{on } \partial\Omega. \end{cases}$$

**Def 27.11** : The Dirichlet BV Problem is to solve

$$\begin{cases} \Delta u = g & \text{on } \Omega \\ u|_{\partial\Omega} = 0 & \text{on } \partial\Omega. \end{cases}$$

## 27.iv) : Some functional analysis

**Lemma 27.11** : If $A : X \to Y$ is a map on inner product spaces (not nec. finite-dim) then

$$(\text{Range } A)^{\perp} = \text{Ker } A^*$$

**Proof** : If $v \perp Ax \;\; \forall x$, then

$$0 = \langle Ax, v \rangle = \langle x, A^* v \rangle \;\; \forall \; x, \text{ in particular } A^* v = x$$
$$\text{so } \; A^* v = 0. \qquad \square$$

**Lemma 27.12** : If $\{f_n\} : \Omega \to \mathbb{R}$ is a sequence on $\Omega$ compact w/

(Rellich's Lemma)

$$\int_\Omega |\nabla f_n|^2 \, dV < C \quad , \quad \int |f_n|^2 \, dV < C.$$

then $\exists$ a subsequence $f_{n_k} \to f$ w/ $\int_\Omega |f|^2 dV < \infty$.

st
$$\int_\Omega |f_{n_k} - f|^2 \, dV \longrightarrow 0.$$

**Proof** : Show $f_n$ is approximately equicontinuous + Arzela-Ascoli.

## 27.v) The Dirichlet problem

**Thm 27.13** : For all $f : \Omega \to \mathbb{R}$ with $\int_\Omega |f|^2 < \infty$, $\exists !\; u : \Omega \to \mathbb{R}$ such that

$$\begin{cases} \Delta u = f \\ u|_{\partial\Omega} = 0. \end{cases}$$

~~**Proof** : Consider $H_0 = \{ f \mid u \text{ st } u|_{\partial\Omega} = 0 \}$.~~

~~**Lemma** (Poincaré) : If $u \in H_0$, then~~

$$\cancel{\int |u|^2 dV < \int |\nabla u|^2 dV}$$

# Lecture 28 The Dirichlet on domains.

Recall

**Thm 27.13**: For all $f : \Omega \to \mathbb{R}$ with $\int_\Omega |f|^2 < \infty$, $\exists !\, u : \Omega \to \mathbb{R}$ with

$$\begin{cases} \Delta u = f \\ u|_{\partial\Omega} = 0 \end{cases}$$

**Proof Idea**: Integration by parts

① 
$$\int_\Omega \langle u, \Delta u \rangle dV = \int_\Omega \nabla \cdot \langle u, \nabla u \rangle + \int_\Omega |\nabla u|^2$$

$$= \int_\Omega |\nabla u|^2 + \int_{\partial\Omega} \langle u, \nabla_\nu u \rangle$$

② If $u|_{\partial\Omega} = 0$

$$\int_\Omega \langle u, \Delta v \rangle = \int_\Omega \langle \nabla u, \nabla v \rangle = \int_\Omega \langle \Delta u, v \rangle + \int_{\partial\Omega} \langle \nabla_\nu u, v \rangle$$

Almost self-adjoint.

**Proof of Theorem**: Consider $u \longmapsto \Delta u$.

$$\int_\Omega \langle u, \Delta u \rangle = \int_\Omega |\nabla u|^2.$$

**Lemma 28.1** (Poincare Inequality): $\exists C_\Omega$ such that for $u|_{\partial\Omega} = 0$,

$$\int_\Omega |u|^2 dV \leq C_\Omega \int |\nabla u|^2 dV.$$

Therefore,

$$\frac{1}{2C_\Omega}\int |u|^2\,dV + \frac{1}{2}\int |\nabla u|^2 \le \int_\Omega \langle u, \Delta u\rangle\,dV$$

Young's Inequality
$ab \le \frac{|a|^2}{2} + \frac{|b|^2}{2}$

$ab \le \frac{|a|^2}{2c} + \frac{c|b|^2}{2}$

$$\le 2C_\Omega \int |\Delta u|^2 + \frac{1}{4C_\Omega}\int |u|^2$$

absorb

$$\int |u|^2 + \int |\nabla u|^2 \le C_2 \int |\Delta u|^2$$

Therefore, if $f = \Delta u = 0$, $u = 0$, so $u \mapsto \Delta u$ is injective.

Surjectivity

If $\forall u$,

$$0 = \int \langle \Delta u, v\rangle = \int_\Omega \langle u, \Delta v\rangle + \int_{\partial\Omega} \langle \nabla u, v\rangle$$

Take $u$ w/ $\nabla u|_{\partial\Omega} = 0$, $\Rightarrow$ $\Delta v = 0$ $\forall x \in \Omega$.

next take $\nabla u$ varying $\Rightarrow$ $v|_{\partial\Omega} = 0$.

By injectivity, $v = 0$ so also surj. $\square$

Sidenote (I'm glossing over closed range)

Proof (of Poincaré) Suppose not. Then $\exists u_n$ w/ $\forall n$ $\int |u_n|^2 \ge N \int |\nabla u_n|^2$

ie if $\int |u_n|^2 = 1$, $\int |\nabla u_n|^2 < \frac{1}{n}$. By Rellich, $\exists u$ st $u_n \to u$ and $\int |u|^2 = 1$.

In fact, can show $\nabla u$ exists and $\nabla u_n \to \nabla u = 0$. So $u$ is const.

But $u|_{\partial\Omega} = 0$ $\rightarrow\leftarrow$.

## 28.ii The Dirichlet Problem

Solve
$$\begin{cases} \Delta u = 0 & \text{on } \Omega \\ u|_{\partial\Omega} = g & \text{on } \partial\Omega \end{cases}$$

Ex 28.2: If $\Omega = \mathbb{D}$, write $g = \operatorname{Re}\left(\sum_{k\in\mathbb{Z}} c_k e^{ik\theta}\right)$ $\quad c_k \in \mathbb{C}$

with $\sum |c_k|^2 = \int |g|^2 < \infty$.

Take $u = \cancel{\ }\left(\sum_{k=-\infty}^{\infty} c_k z^k\right)$. Then $\bar\partial u = 0 \Rightarrow \Delta(\operatorname{Re} u) = 0$.

$\operatorname{Re} u|_{\partial\Omega} = \operatorname{Re}(\quad) = g$ ✓.

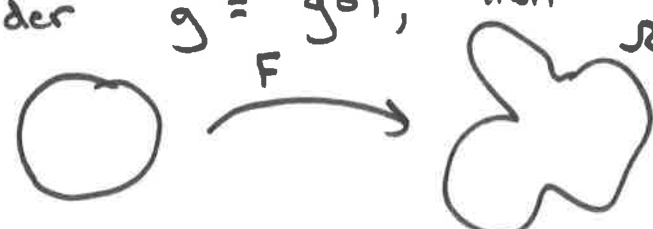
And $\int |u|^2 = \int |\sum c_k z^k|^2 \le \sum |c_k|^2 \int |z^k|^2 \le \sum_{k=\infty}^{\infty} \frac{|c_k|^2}{2k+2} < \infty$.

**Theorem 28.3** (Solvability of Dirichlet Problem): Suppose that $\Omega$ is simply connected, and $\partial\Omega$ is $C^2$. Then $\exists!$ solution $u$ of
$$\begin{cases} \Delta u = 0 \\ u|_{\partial\Omega} = g \text{ on } \partial\Omega \end{cases} \qquad \int_{\partial\Omega} |g|^2 < \infty.$$
with $\int |u|^2 < \infty$.

**Proof**: Carathéodory's Extension:

**Thm 28.4**: The Riemann map $F : \mathbb{D} \to \Omega$ extends to a ~~homeomorphism~~ $C^\infty$ map of $\bar F : \partial\mathbb{D} \to \partial\Omega$, provided $\partial\Omega$ is $C^\infty$.

Therefore, consider $\tilde g = g \circ F$, then $\int_{\partial\mathbb{D}} |\tilde g|^2 < \infty$, so




By example 28.2, $\exists!\ u : \mathbb{D} \to \mathbb{R}$ st $\begin{cases} \Delta \tilde u = 0 \\ u|_{\partial\mathbb{D}} = \tilde g. \end{cases}$

Let $\tilde{U} = \{$holomorphic w/ $\mathrm{Re}\,\tilde{U} = \tilde{u}\}$. Take

$$u = \mathrm{Re}(\tilde{U} \circ F^{-1}).$$

Then $\tilde{U} \circ F^{-1}$ is holomorphic, so $u$ is harmonic. And $u|_{\partial\Omega} = g$ by construction. $\square$

<u>Corollary 28.5</u> (Full Dirichlet Problem)

If $\Omega \subseteq \mathbb{C}$ is smooth and simply connected, $\exists!\, u$ solving

$$\begin{cases} \Delta u = f & \text{on } \Omega \\ u|_{\partial\Omega} = g & \text{on } \partial\Omega. \end{cases}$$

<u>Proof</u>: Let $u_1$ solve $\begin{cases} \Delta u = f \\ u|_{\partial\Omega} = 0 \end{cases}$

$u_2$ solve $\begin{cases} \Delta u = 0 \\ u|_{\partial\Omega} = g, \end{cases}$ set $u = u_1 + u_2$.

<u>Rem 28.6</u>: It is NOT true that one can solve

$$\begin{cases} \bar{\partial} f = g \\ f|_{\partial\Omega} = h \end{cases}$$

this is overdetermined because $\bar{\partial}$ is first order.

$F = \frac{\partial^2}{\partial x^2} + b\frac{\partial}{\partial x}$ on $\mathbb{R}$

$\Rightarrow$ bd conditions = order

1st order = "half bd condition".

boundary value problems for first order operators are called Atiyah-Patodi-Singer boundary value problems (APS, 1975).

# Lecture 29 | The Poisson Kernel + Fundamental Solution

We have shown that

$$\mathcal{H}(\mathbb{D}) \xrightarrow{(\Delta, |_{\partial\mathbb{D}})} \mathcal{C}^\infty(S^1)$$

harmonic functions $\quad u \longrightarrow f$ st $\begin{cases} \Delta u = 0 & \text{on } \mathbb{D} \\ u|_{\partial D} = f & \text{on } \partial\mathbb{D} \end{cases}$

is an isomorphism.

Question 29.1: Can we describe the Inverse "Poisson Operator"

$$\mathcal{C}^\infty(S^1) \longrightarrow \mathcal{H}(\mathbb{D}) \subseteq \mathcal{C}(\mathbb{D})$$

explicity?

## 29.i The idea of an integral Kernel

Fact 29.2: $\Delta u$ is linear, so is $|_{\partial\mathbb{D}}$. Therefore if

$$\begin{cases} \Delta u_1 = 0 \\ u_1|_{\partial D} = f_1 \end{cases} \qquad \begin{cases} \Delta u_2 = 0 \\ u_2|_{\partial D} = f_2 \end{cases}$$

$$\Rightarrow \quad \begin{cases} \Delta(u_1+u_2) = 0 \\ u_1+u_2|_{\partial D} = f_1+f_2. \end{cases}$$

Idea 29.3: Decompose $f$ into an infinite sum of pieces with Known solutions.

Version 1): Spectrally, write $f = \sum_{k=-\infty}^{\infty} c_k e^{ik\theta}$

Version 2): locally

Def 29.4: the Dirac delta "function" is the map

$$\int \delta(x-x_0) : \mathcal{C}^0(S^1) \longrightarrow \mathbb{R}$$

by $\quad f \longmapsto \int f(x)\delta(x-x_0)dx = f(x_0).$

$\Rightarrow$ The identity operator may be written

$$f(x) = \int_{S^1} f(y)\delta(y-x)dy = \text{"}\sum_y f_y \delta(y-x)\text{"}.$$

Suppose $P_{\theta_0}(x)$ solves or $P(r, \theta - \theta_0)$

$$\begin{cases} \Delta P_{\theta_0}(x) = 0 \\ P_{\theta_0}(x)\Big|_{\partial D} = \delta(\theta) \in (S^1). \end{cases}$$

Lemma 29.5 : Formally, if we can find $P_\theta(x)$ as above, then

$$u = \int_{S^1} f(\theta') P(r, \theta - \theta')\, d\theta'$$

solves

$$\begin{cases} \Delta u = 0 \\ u|_{\partial D} = f. \end{cases}$$

Proof :

$$\Delta_\theta u = \Delta_\theta \int_{S^1} f(\theta') P(r, \theta - \theta')\, d\theta'$$

$$= \int_{S^1} f(\theta') \Delta_\theta P(r, \theta - \theta')\, d\theta'$$

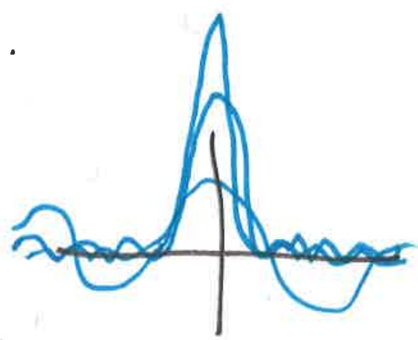
$$= 0$$

$$u|_{S^1} = \int_{S^1} f(\theta') \delta(\theta - \theta')\, d\theta' = f(\theta). \quad \square$$

29.ii) : The Formula for the Poisson Kernel.

Lemma 29.6 : $\delta(\theta) = \frac{1}{2\pi} \sum_{k \in \mathbb{Z}} e^{ik\theta}$.

in Fourier series



*whatever = means

IF $f(\theta) = \sum c_k e^{ik\theta}$ in Fourier series

Proof :

$$f(0) = \frac{1}{2\pi} \int f(\theta) \delta(\theta)\, d\theta$$

$$\sum_{k \in \mathbb{Z}} c_k \qquad = \frac{1}{2\pi} \int \sum_k c_k e^{ik\theta} \cdot \sum_\ell d_\ell e^{i\ell\theta} \implies f(0) = \sum c_k$$

$$= \sum_{k = -\ell \in \mathbb{Z}} c_k d_\ell \qquad \forall c_k \text{ so } d_\ell = 1. \quad \square$$

**Prop 29.7**: The Poisson Kernel is given by

$$P(r,\theta) = \frac{1}{2\pi} \frac{1-r^2}{1-2r\cos\theta + r^2}$$

**"Proof"**: By previous example, solution of

$$\begin{cases} \Delta u = 0 \\ u\big|_{S^1} = \sum_{k\in\mathbb{Z}} c_k e^{ik\theta} \end{cases}$$

is $u = \sum c_k z^k$.

For $\delta = \sum_{k\in\mathbb{Z}} e^{ik\theta}$, $\begin{cases} \Delta P_\theta(r,\theta) = 0 \\ P_\theta\big|_{S^1} = \delta(\theta) \end{cases}$

$$\Rightarrow P(r,\theta) = \frac{1}{2\pi}\sum_k r^k e^{ik\theta}$$

$$= \frac{1}{2\pi}\left[1 + \sum_{k\in\mathbb{N}} r^k e^{ik\theta} + r^k e^{-ik\theta}\right]$$

$$= \frac{1}{2\pi}\left[-1 + \frac{1}{1-re^{i\theta}} + \frac{1}{1-re^{-i\theta}}\right]$$

$$= \frac{1}{2\pi}\left[\frac{1-re^{-i\theta} + 1 - re^{i\theta}}{1-2r\cos\theta + r^2} - \frac{1-2r\cos\theta + r^2}{1-2r\cos\theta + r^2}\right]$$

$$= \frac{1}{2\pi}\left[\frac{1-r^2}{1-2r\cos\theta + r^2}\right]$$

□

**Prop 29.8** (Rigorous Version).

Define $J_R(f) : S^1 \to \mathbb{R}$

$$J_R(f) = \int_{S^1} f(\theta) P(R, \theta - \theta')\, d\theta'$$

Then $J_R(f) \to f$ uniformly on $S^1$.

**Proof**: $J_R(f) = \int_{S^1} f(\theta-\theta')p_R(\theta')d\theta'$

For fixed $\theta \neq 0$, $|P_r(\theta)| \leq \frac{1}{2\pi}\frac{1-r^2}{1+r^2+\delta} \leq \left|\frac{1-r^2}{c}\right| \to 0$ as $r \mapsto 1$

so take $\varepsilon > 0$ and $\delta$ st for $|\theta'| > \delta$,

$$\int_{|\theta'|>\delta} |P_r(\theta')|\,d\theta' < \varepsilon.$$

So

$$\int_{|\theta'|\leq\delta} |P_r(\theta')| \geq 1-\varepsilon.$$

Then

$$J_R(f) = \int_0^{2\pi} f(\theta-\theta') P_R(\theta')d\theta'$$

$$= \int_{-\delta}^{\delta} f(\theta-\theta') P_R(\theta')d\theta' + \underbrace{\int_{|\theta'|\geq\delta} f(\theta-\theta') P_R(\theta')d\theta'}_{\varepsilon \cdot \sup|f|.}$$

$$= \int_{-\delta}^{\delta} [f(\theta) - f(\theta) + f(\theta-\theta')] P_R(\theta')\,d\theta$$

$$= f(\theta)\int_{-\delta}^{\delta} P_R(\theta')d\theta' + \underbrace{\sup|f(\theta-\theta')-f(\theta)|}_{\to 0 \text{ w/ } \delta \text{ by continuity}} \cdot 1$$

$$= f(\theta) + \varepsilon.$$

$\square$

**Rem 29.9**: All linear PDEs have an integral Kernel / Schwartz Kernel / Fundamental Solution / Greens function

$$\begin{cases} Lu_0 = \delta(x_0) \text{ on } \Omega \\ u_0|_{\delta\Omega} = 0 \end{cases} \quad \text{and} \quad \begin{cases} Lu_0 = 0 \\ u_0|_{\partial D} = \delta(y_0) \text{ on } \delta\Omega \end{cases}$$

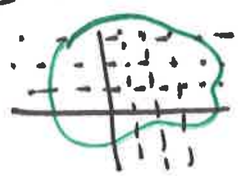
Poisson Kernel

See Math 173/175 :) + Math 205.

# Lecture 24 The Riemann Sphere and $PSL(2,\mathbb{C})$.

## 24.i) The sphere as a surface

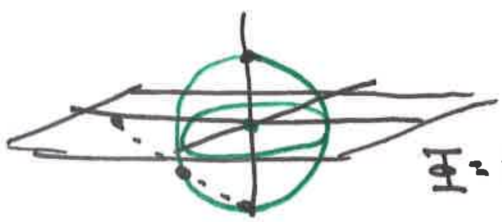
- Every open domain $\Omega \subseteq \mathbb{C}$ comes with a coordinate system $z = x+iy$. The sphere $S^2 = \{(x,y,z) \mid x^2+y^2+z^2=1\} \subseteq \mathbb{R}^3$ has no global (single) coordinate system.



**Def 24.1**: The stereographic coordinate systems at $N^+=(0,0,1)$, $N^-=(0,0,-1)$ are homeomorphisms

$$U_\pm : \mathbb{R}^2 \longrightarrow S^2 \setminus N^\mp$$

given by $(x,y) \longmapsto \left( \frac{2x}{1+x^2+y^2}, \pm\frac{2y}{1+x^2+y^2}, \pm\frac{(-1+x^2+y^2)}{1+x^2+y^2} \right)$



The coordinate transition function is the diffeomorphism

$$\Phi = U_+ \circ U_-^{-1} : \mathbb{R}^2 \setminus \{0\} \longrightarrow \mathbb{R}^2 \setminus \{0\}$$

**Lemma 24.2**: $\Phi$ is holomorphic, in fact it is $z \to \frac{1}{z}$.

**Proof**: $z = x+iy \longmapsto \left( \frac{2z}{1+|z|^2}, \frac{-1+|z|^2}{1+|z|^2} \right)$

$w = u+iv \longmapsto \left( \frac{2\bar{w}}{1+|w|^2}, \frac{|w|^2-1}{1+|w|^2} \right)$

Suffices to check $z$, $\underset{w=\frac{1}{z}}{\cancel{\tfrac{1}{z}}}$ map to same point, $w = \frac{1}{z} \longmapsto \left( \frac{2\overline{\frac{1}{z}}}{1+\frac{1}{|z|^2}}, \frac{1-\frac{1}{|z|^2}}{1+\frac{1}{|z|^2}} \right)$

$$= \left( \frac{2|z|^2/\bar{z}}{|z|^2+1}, \frac{|z|^2-1}{1+|z|^2} \right) \quad \square.$$

**Def 24.3**: a holomorphic (resp. meromorphic) map on the Riemann sphere is one that is holomorphic (meromorphic) in each coordinate chart. Same for a $\overset{\text{hol.}}{\text{map}}$ $\longrightarrow$ sphere.

**Rem/Ex 24.4**: A meromorphic function $\Omega \to \mathbb{C}$ is just a holomorphic function to the sphere.

**Lemma 24.5** (Liouville) There are no non-constant holomorphic functions on the Riemann Sphere.

**Proof**: Cont + Compact $\Rightarrow$ bounded.

**Prop 24.6** (Argument Principle) If $h$ is meromorphic on Riemann sphere,

$$\#\text{zeros} - \#\text{poles} = 0, \quad \text{counted w/ multiplicity.}$$

**Def 24.7**: The complex projective space of 1-dim is

$$\mathbb{P}^1 = \mathbb{CP}^1 = \{ L \subseteq \mathbb{C}^2 \mid L \text{ is a complex, 1-dim. vectorspace} \}.$$

$$= \mathbb{C}^2 \setminus \{0\} / \mathbb{C}^\times \quad \text{where } \mathbb{C}^\times \text{ acts by multiplication.}$$

**Prop 24.8**: The Riemann sphere and $\mathbb{CP}^1$ are biholomorphic

**Proof**: Points are described by proj. coordinate equivalence classes

$$[z:w] \text{ w/ } (z,w) \in \mathbb{C}^2 \setminus \{0\} \text{ w/ } [z:w] = [\lambda z : \lambda w].$$

Provided, $z, w \neq 0$ then

$$[\tfrac{z}{w} : 1] = [z:w] = [1 : \tfrac{w}{z}]$$

Thus $z \longmapsto [z:1]$ and $w \longmapsto [1:w]$ are charts each omitting a single point w/ transition $z \to \frac{1}{z}$ $\square$.

**Rem 24.9**: $\mathbb{CP}^1$ is the simplest example of a moduli space, in which each point is itself a geometric object.

## 24.ii) PSL(2, $\mathbb{C}$)

Let $GL(2,\mathbb{C}) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \middle| \ a,b,c,d \in \mathbb{C},\ \det \neq 0 \right\}$

**Lemma 24.10**: The center is $Z = \{ A \mid AB = BA \ \ \forall B \in GL(2,\mathbb{C}) \}$

$$= \left\{ \begin{pmatrix} \lambda & \\ & \lambda \end{pmatrix} \middle| \ \lambda \in \mathbb{C}^\times \right\}.$$

**Def 24.11**
The Projective ~~general~~ special linear group is

$$PGL(2,\mathbb{C}) = GL(2,\mathbb{C})/\mathbb{Z} \simeq SL(2,\mathbb{C})/\{\pm 1\} = PSL(2,\mathbb{C}).$$

**Remark**: $GL(2,\mathbb{C}) \simeq \mathbb{C}^4$ geometrically,
$SL(2,\mathbb{C}) = \{ad - bc = 1\} \subseteq \mathbb{C}^4$ is a smooth manifold of $\dim_{\mathbb{C}} = 3$, $\dim_{\mathbb{R}} = 6$.
$\{\pm 1\}$ acts discretely w/o fixed points so $PSL(2,\mathbb{C})$ is also a manifold.

**Def 24.12**: the Lie algebra of a Lie group is the tangent space at the Id.

$$\mathfrak{sl}(2,\mathbb{C}) = \mathrm{Ker}_{Id}\{d(ad-bc) \to \mathbb{C}\}.$$

It can be checked that

$$\mathfrak{sl}(2,\mathbb{C}) = \mathfrak{su}(2) \oplus i\,\mathfrak{su}(2) \qquad \mathfrak{su}(2) = \left( \begin{array}{c} \text{skew-hermitian traceless} \\ \begin{pmatrix} -i & \\ & i \end{pmatrix} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix} \end{array} \right).$$

**Def 24.13**: The automorphism group of $\mathbb{P}^1$ is

$$\mathrm{Aut}(\mathbb{P}^1) = \{\Phi : \mathbb{P}^1 \to \mathbb{P}^1 \text{ biholomorphic}\}.$$

a priori, it has, say, the $C^0$ topology

**Remark**: a priori, it's not even clear $\mathrm{Aut}(\mathbb{P}^1)$ is finite-dimensional.
$\mathrm{Diff}(\mathbb{P}^1)$ is $\infty$-dim, but

$$\mathfrak{diff}(\mathbb{P}^1) = T_{Id}\,\mathrm{Diff}(\mathbb{P}^1)$$
$$= \{\text{Vector fields on } \mathbb{P}^1\} \supseteq (\text{hol. vector fields}) \simeq \mathfrak{aut}(\mathbb{P}^1).$$
dim 3.

**Prop 24.14**: The map $PSL(2,\mathbb{C}) \to \mathrm{Aut}(\mathbb{P}^1)$ by

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \longmapsto \frac{az+b}{cz+d}$$

is a diffeomorphism.

**Proof**: Suppose that $f : \mathbb{P}^1 \to \mathbb{P}^1$ is a biholomorphism. Then $f$ is a meromorphic function on $\mathbb{C}_\infty$ with a single zero and pole (possible at infinity).
Thus $f$ = rational function $= \frac{p(z)}{q(z)}$ and must be degree 1. This shows surjectivity. It's easy to check only $-1$ goes to Id. $\square$

Three distinguished subgroups:

$$SU(2) = \left\{ \begin{pmatrix} a & -b \\ \bar{b} & \bar{a} \end{pmatrix} \,\middle|\, a,b \in \mathbb{C},\ |\alpha|^2 + |\beta|^2 = 1 \right\} \simeq S^3$$

$$PSU(1,1) = \{ \quad | \quad |\alpha|^2 - |\beta|^2 = 1 \} \simeq S^1 \times \mathbb{R}^2$$

$$PSL(2,\mathbb{R}) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \,\middle|\, a,b,c,d \in \mathbb{R},\ \det = 1 \right\} \simeq S^1 \times \mathbb{R}^2$$

**Lemma 24.15**: Every $u \in PSL(2,\mathbb{C})$ is conjugate to one of the following

(i) $\pm \mathrm{Id}$

(ii) $\pm \begin{pmatrix} \lambda & \\ & \lambda^{-1} \end{pmatrix}$ $\lambda \neq 1$. (semisimple)

(iii) $\pm \begin{pmatrix} 1 & \pm 1 \\ & 1 \end{pmatrix}$ (unipotent case)

**Proof**: This is Jordan normal form (conjugation can be taken in $SL(2,\mathbb{C})$) (not $\pm$ are same in quotient).

## 24.iii) Subgroups of $PSL(2,\mathbb{R})$

**Def 24.16**: $\mathrm{Tr}(u), \det(u) \in \mathbb{R}$. An element $u \in PSL(2,\mathbb{R})$ is

| $\mathrm{Tr}(u) < 4\det(u)$ | $\mathrm{Tr}(u) = 4\det(u)$ | $\mathrm{Tr}(u) > 4\det(u)$ |
|---|---|---|
| (elliptic) | (parabolic) | hyperbolic |
| $\lambda_i$ = conjugates in $\mathbb{C}$, | $\lambda_i$ : mult 2 real | $\lambda_i$ : distinct reals. |
| $\left\{ \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \middle| \theta \in [0,2\pi) \right\}$ | $\left\{ \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix} \middle| b \in \mathbb{R} \right\}$ | $\left\{ \begin{pmatrix} e^t & \\ & e^{-t} \end{pmatrix} \middle| t > 0 \right\}$ |

**Prop 24.17**: Each elliptic / parabolic / hyperbolic element is conjugate in $PSL(2,\mathbb{R})$ to

- $\left\{ \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \middle| \theta \in (0,2\pi) \right.$
- $\begin{pmatrix} 1 & \pm 1 \\ & 1 \end{pmatrix}$
- $\begin{pmatrix} e^t & \\ & e^{-t} \end{pmatrix}$ $t > 0$.

**Proof**: Real Jordan normal form.

**Prop 24.18**:
- Elliptic iff fixed points $z_0, \bar{z}_0$ w/ $z_0 \in \mathbb{H}$.
- Parabolic iff unique fixed point on $\mathbb{R} \cup \{\infty\}$.
- hyperbolic iff two distinct fixed points on $\mathbb{R} \cup \{\infty\}$.

**Proof**:

Parabolic fixes $\infty$.

fixes $0, \infty$

fixes $i, -i$.

$\square$

# Lecture 25 Hyperbolic Geometry + Fuchsian Groups

## 25.i) The hyperbolic metric

Each domain $\Omega \subseteq \mathbb{C}$ inherits a distance $d(z,w) = \sqrt{(z-w)\cdot(z-w)}$

$$= \inf_{\gamma: z \to w} \int_\gamma |\dot\gamma|^2 = \inf_{\gamma: z\to w} \int_\gamma \dot\gamma^T \mathrm{Id}\, \dot\gamma$$

**Def 25.1**: A (Riemannian) metric on an open domain $\Omega \subseteq \mathbb{C}$ is a smooth function

$$g : \Omega \longrightarrow gl(2,\mathbb{C})$$

into the symmetric, positive definite matrices. (More generally, e.g. on $\mathbb{CP}^1$ it is one in coordinates intertwined by coordinate change).

**Ex 25.2**: $g_0 = \mathrm{Id}$ is called the Euclidean metric.

**Ex 25.3**: Two metrics are conformal if $g_1 = e^u g_2$ for $u : \Omega \to \mathbb{R}$.

$f : \Omega_1 \to \Omega_2$ holomorphic then pullback

$$f^* g = df^T \cdot g\, df$$

is conformal for $g_0 = g$.

**Def 25.4**: A map $\Phi : (\Omega_1, g_1) \to (\Omega_2, g_2)$ is an <u>isometry</u> if

$$\Phi^* g_2 = g_1$$

**Rem 25.5**: A Riemannian metric makes $\Omega$ into a metric space by

$$\mathrm{dist}(z,w) = \inf_{\gamma: z\to w} \int_\gamma \dot\gamma^T g\, \dot\gamma$$

**Def 25.6**: The hyperbolic metric on $\mathbb{H} = \{\mathrm{Im}\, z > 0\}$ is

$$h = \frac{1}{\mathrm{Im}(z)^2}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

**Prop 25.7**: The action of mobius transformations

$$PSL(2,\mathbb{R}) \longrightarrow \mathrm{Iso}(\mathbb{H}, h)$$

is by isometries.

**Proof**: Suffices to show for $M_b = \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix}$ $M_\lambda = \begin{pmatrix} \lambda & \\ & \lambda^{-1} \end{pmatrix}$ $M_\theta = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$

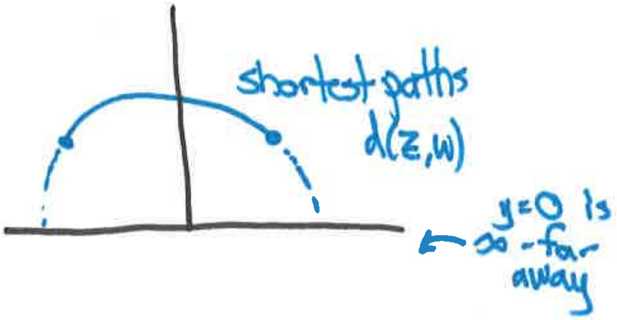
① Intuitively obvious by translation, ③ Exercise.

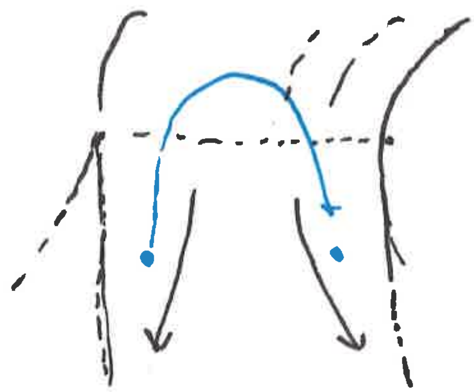
②


$$\frac{1}{y^2}\begin{pmatrix} 1 & \\ & 1 \end{pmatrix} = df^T \cdot \frac{\mathrm{Id}}{(\lambda^2 y)^2} df$$

$$= \lambda^2 \frac{1}{\lambda^4 y^2}\begin{pmatrix} 1 & \\ & 1 \end{pmatrix} \cdot \lambda^2\, \mathrm{Id} \checkmark$$

Hyperbolic geodesics



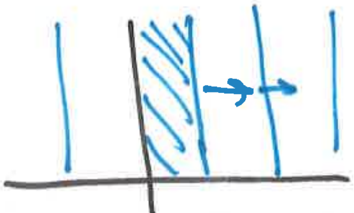




25.ii) Group Actions : Any subgroup $\Gamma \subseteq PSL(2,\mathbb{R})$ acts by

$$(M ; z) \longmapsto M(z)$$
$$\Gamma \times \mathbb{H} \longrightarrow \mathbb{H}$$

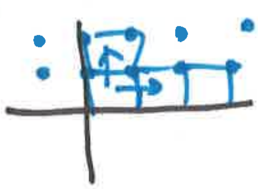
Def 25.8
The quotient is $\mathbb{H}/\Gamma = \{\Gamma \cdot z \mid \Gamma z$ is a coset of the group action$\}$.

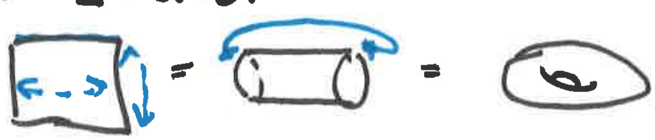
Ex 25.9 : $M = \begin{pmatrix} 1 & b \\ & 1 \end{pmatrix}$ acts by translation



so $\mathbb{H}/\Gamma \cong$  $= S^1 \times \mathbb{R}^{>0}$.

Ex 25.10 : $\mathbb{Z}^2$ acts on $\mathbb{C}$ by (
$$((a,b), z) \longmapsto z + a + bi$$



$\mathbb{C}/\mathbb{Z}^2 =$  $S^1 \times S^1$

Def 25.11 : A group $\Gamma \curvearrowright \mathbb{H}$ by isometries is called properly discontinuously if $\forall z \in \mathbb{H}$ the orbit
$$\Gamma z = \{\gamma z \mid \gamma \in \Gamma\}$$
is locally finite i.e. $\exists U \ni z$ st $\gamma U \cap U = \emptyset \ \forall \gamma \in \Gamma$.

**Thm 25.12**: A subgroup $\Gamma \subseteq PSL(2,\mathbb{R})$ acts properly discontinuously on $\mathbb{H}$ if and only if it is discrete. Such a $\Gamma$ is called <u>Fuchsian</u>.

<u>Proof</u>: $\Rightarrow$ Suppose $\Gamma$ is not discrete, Then $\exists \gamma_n \in \Gamma$ st $\gamma_n \to \gamma \in \Gamma$.
Consider $\gamma^{-1}\gamma_n \to \mathrm{Id} \in \Gamma$. Either infinitely many fix $i$ or not.

I) If $\infty$-many do not fix $i$ then $\gamma^{-1}\gamma_n \cdot i \to i$ accumulates $\Rightarrow \not$ prop disc.

II) If $\infty$-many fix $i$ $\gamma^{-1}\gamma_n$ are elliptic fixing $i$ and $\to \mathrm{Id}$.
In this case $\gamma^{-1}\gamma_n \cdot 2i \to 2i$ accumulates. $\to \not$.

$\Leftarrow$ Suppose $\Gamma$ is discrete. Let $K \subseteq \mathbb{H}$ be compact. It suffices to show that

$$|\Gamma \cdot z \cap K| < \infty$$

is finite.

could have non-unique $g \in \Gamma$.

$$|\Gamma z \cap K| \leq \Gamma \cap \{g \in PSL(2,\mathbb{R}) \mid gz \in K\}$$

Claim the latter is compact ($\Rightarrow$ finite since $\Gamma$ discrete $\Rightarrow$ closed)

- Thus let $g_n \in PSL(2,\mathbb{R})$, $g_n \to g \in PSL(2,\mathbb{R})$.

$$\underset{\in K}{g_n z} \to gz \Rightarrow gz \in K \text{ b/c } K \text{ is closed.}$$

- For bounded suffices to show that $a,b,c,d$ st $\frac{az+b}{cz+d} \in K$ are bounded. Since $K$ is bounded, $\exists C$ s.t. $K \subseteq \{|z| < C \cap \mathrm{Im}\, z > \frac{1}{C}\}$

$$|az+b| \leq C|cz+d|.$$

but

$$\frac{1}{C} < \mathrm{Im}\left(\frac{az+b}{cz+d}\right) = \mathrm{Im}\left(\frac{(az+b)\overline{(cz+d)}}{|cz+d|^2}\right) \overset{a,b,c,d \in \mathbb{R}}{=} \frac{\mathrm{Im}(z)(ad-bc)}{|cz+d|^2} = \frac{\mathrm{Im}(z)}{|cz+d|^2}$$

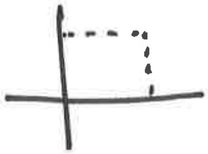
$\Rightarrow$ c,d bounded $\Rightarrow$ a,b bounded. $\square$

**Def 25.13**: A subset (open) $F \subseteq \mathbb{H}$ is said to be a <u>Fundamental domain</u> for the action of $\Gamma \subseteq PSL(2,\mathbb{R})$ Fuchsian $\curvearrowright \mathbb{H}$ if

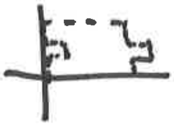
$g F \cap g' F = \emptyset \ \forall g \neq g'$, and $\bigcup_{\Gamma} g \cdot \overline{F} = \mathbb{H}$ cover.

$\Rightarrow gF$ tesselate the plane.

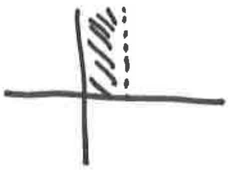
**Ex 25.14**



is fundamental domain of $\Lambda = \mathbb{Z}^2 \curvearrowright \mathbb{C}$.



fundamental domain is not unique



for $\begin{pmatrix} 1 & b \\ & 1 \end{pmatrix}$.

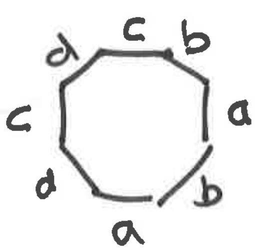
**Def 25.15** : $\Gamma \subseteq PSL(2,\mathbb{R})$ acts freely if it has no fixed points.

**Thm 25.16** : Suppose $\Gamma \subseteq PSL(2,\mathbb{R})$ is Fuchsian w/ free action and fundamental domain has finite area. Then

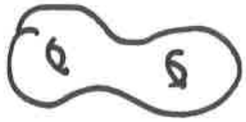
$$\mathbb{H}/\Gamma$$

is a compact, Hausdorff, $2^{nd}$ countable space locally biholomorphic to $\Omega \subseteq \mathbb{H}$ equipped w/ its hyperbolic metric, i.e.

genus g.




associate sides
$\longrightarrow$



**Question 25.17** : How can we see topological information, e.g. genus from $\Gamma$? For what $\Gamma_1, \Gamma_2$ are the surfaces biholomorphic?

Questions like this are the stand of Teichmüller theory.