Lecture 1 | Motivation for ODEs + Definitions

1.i) Introduction + Syllabus

1.ii) Definitions:

Def 1.1: An <u>Ordinary Differential Equation</u> (ODE) is an equality relating an unknown function (of a single variable) to its own derivatives.

Def 1.2 A <u>solution</u> is a function that satisfies the equation everywhere the function is defined.

Ex 1.3: The following are examples of ODEs

i) $\frac{dy}{dx} = 3y(x)$

ii) $\frac{dy}{dx} = y(x)^2 + 1$

iii) $\frac{d^2y}{dx^2} + 3\frac{dy}{dx} + 2y(x) = 0$

iv) $\frac{d}{dx}\left(x^2 \frac{dy}{dx}\right) = x^2 e^{-y(x)}$

v) $\frac{dy}{dt} = -k\sqrt{y(t)}$ for $k \in \mathbb{R}$ a fixed real number, called a <u>constant</u> in the eq.

vi) $\frac{d^3y}{dx^3} - y(x)\left(\frac{dy}{dx}\right)^2 = 2y(x) + e^{y(x)}$

Def 1.4: The highest derivative that appears is called the <u>order</u> of the ODE (above, orders 1, 1, 2, 2, 1, 3).

Notation 1.5: The equation doesn't depend on the notation used: The unknown can be written e.g. $y(x), f(x), f(t), x(t), u(x), \Psi(x)$
$A(z), \Xi(\xi), \ldots$

Notation 1.6: We will use "prime" $y(x), y'(x), y''(x), y^{(3)}(x), \ldots$ and
"Leibniz notation" $\frac{dy}{dx}, \frac{d^2y}{dx^2}, \ldots$ interchangeably.

Ex 1.7: Some solutions

i) The function $y(x) = e^{3x}$ solves i)

$$\frac{d}{dx}y(x) = \frac{d}{dx}e^{3x} = 3e^{3x} = 3y(x)$$ ← equality holds for <u>every</u> x i.e. its an equality of functions.

ii) So does $y(x) = 5e^{3x}$; $\frac{dy}{dx} = \frac{d}{dx}5e^{3x} = 5 \cdot 3e^{3x} = \cancel{5}\, 3y(x)$ ✓

iii) $y(x) = \tan(x)$ solves ii)

$$\frac{d}{dx}\tan(x) = \sec^2(x) = 1+\tan^2(x) = 1+y(x)^2 \checkmark$$

↳ $\cos^2+\sin^2=1$
$1+\tan^2=\sec^2$

iv) $y(x) = 5\tan(x)$ does <u>not</u> solve ii)

$$\frac{d}{dx}5\tan(x) = 5\sec^2(x) = 5+5\tan^2(x)$$
$$\neq$$
$$1+y(x)^2 = 1+25\tan^2(x)$$

1.(iii) Linear ODEs + <u>Superposition</u>

Observed that i) and ii) of Example 1.3 behave differently insofar as multiples of solutions are still solutions for i) but not ii). This is a property of linear ODEs.

↙ $y^2, \left(\frac{dy}{dx}\right)^2$ do not appear,

<u>Def 1.8</u>: An ODE is said to be <u>linear</u> if it has the form

$$a_k(x)\frac{d^ky}{dx^k} + a_{k-1}(x)\frac{d^{(k-1)}y}{dx^{k-1}} + \dots + a_1(x)\frac{dy}{dx} + a_0(x)y = f(x).$$

for functions $a_0(x), \dots, a_k(x)$. It is additionally said to be <u>homogeneous</u> if $f(x)=0$, and <u>inhomogeneous</u> otherwise.

<u>Ex 1.9</u>: i) $-5y'''(x) + 6y'(x) + 7y = x\sin x$ (Inhomogeneous <u>linear</u> 3rd order)

ii) $y''(x) + y(x)y'(x) = 0$ (homogeneous <u>nonlinear</u> 2nd order)

iii) $y''(x) + xy'(x) = e^x y(x)$ (homogeneous <u>linear</u> 2nd order)

iv) $(y')^2 + xy = 0$ (homogeneous nonlinear 1st order)

v) $y' + k\sqrt{y} = 0$ ( ).

<u>Thm 1.10</u>: If $y_1(x), \dots, y_N(x)$ are solutions of a homogeneous linear ODE then any linear combination

(Principle of Superposition)

$$Y(x) = \sum_{n=1}^{N} c_n y_n(x) = c_1y_1(x) + \dots + c_Ny_N(x) \qquad c_n \in \mathbb{R}$$

is also a solution. i.e. the set of solutions is a vector space.

<u>Ex 1.11</u>: If $y_1(x)$, $y_2(x)$ satisfy $y''+2y'-3y=0$ then so does $Y = c_1y_1+c_2y_2$.

$$Y''+2Y'-3Y = (c_1y_1+c_2y_2)'' + 2(c_1y_1+c_2y_2)' - 3(c_1y_1+c_2y_2)$$
$$= [c_1y_1'' + 2c_1y_1' - 3c_1y_1] + [c_2y_2'' + 2c_2y_2' - 3c_2y_2] = 0.$$

1. iv) Motivation for ODEs

Philosophy 1.12: The ability to solve ODEs allows you to predict future of physical systems'.

$$\left\{\begin{matrix}\text{ODE governing system}\\ \text{+ state at time } t=0\end{matrix}\right\} \xrightarrow{\text{solve ODE}} \left\{\begin{matrix}y(t) \text{ state of}\\ \text{system at future times}\end{matrix}\right\}.$$

Applications 1.13: ODEs and their cousins PDEs arise in

[See Wikipedia "List of named differential equations"]

1) (Mechanical Systems) Newton's law of motion $F = ma$

$$m\frac{d^2x}{dt^2} = F(t,x)$$

is an ODE. Force ODEs have applications in civil, automotive engineering.

2) (Waves) The wave equation governs propagation of waves used in electronics, optics, signal processing, lasers, etc.

3) (Heat + thermodynamics) The heat and Boltzmann Equations govern heat transfer and thermodynamics.

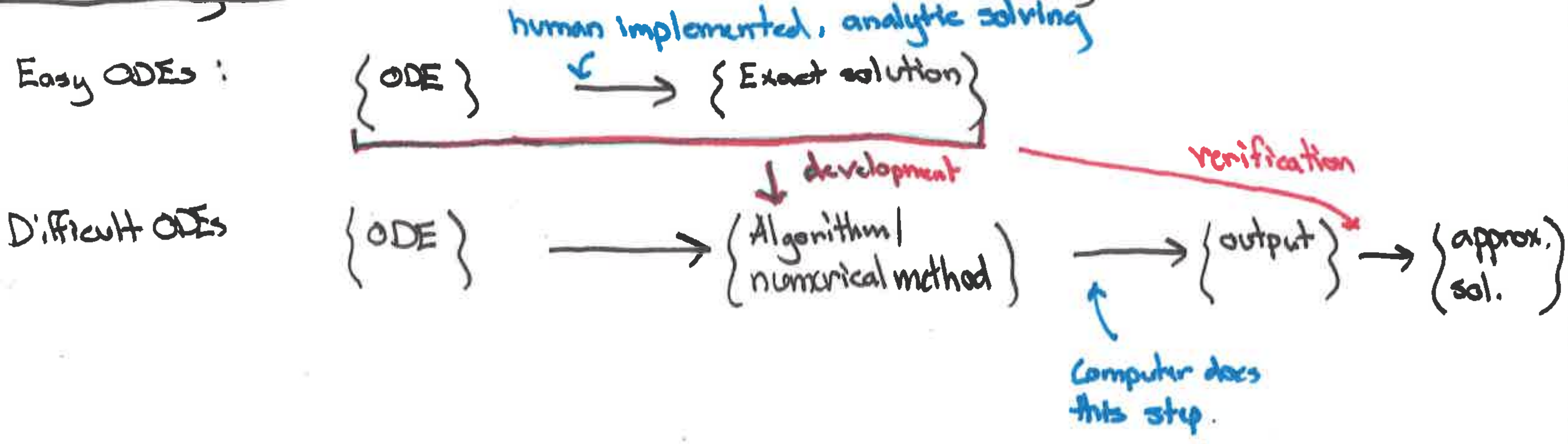
4) (Fluid + Aerodynamics) Governed by the Navier-Stokes equations

5) (Biology) spread of cancer, infectuous diseases, dynamics of neurons, populations, ...

6) (Chemistry) Chemical reactions, phase changes

7) (Economics/Computer Science) Optimization, game theory, etc....

Philosophy 1.14: Modern techniques combine explicit solving + computer modeling



Lecture 2 Autonomous + Separable 1st order ODEs.

2.1 Autonomous 1st Order ODEs

Def 2.1: A 1st order ODE

$f: \mathbb{R}^2 \to \mathbb{R}$

$$\frac{dy}{dt} = f(y(t), t)$$

is said to be autonomous if f depends only on first argument, i.e. t only appears inside $y(t)$.

Ex 2.2:

$$\frac{dy}{dt} = y(t)^2 \qquad \frac{dy}{dt} = \cos(y(t)) + \tan(y(t))$$

$$\frac{dy}{dt} = \log y(t) + y(t)^2 + 3$$

$$\frac{dy}{dt} = \sum_{k=1}^{\infty} \frac{1}{k^2} e^{\cos(e^{-y(t)^3})} + \frac{1}{k^2} \arctan(y(t)).$$

All of these are autonomous.

Ex 2.3 (non-autonomous ODEs)

explicit t-dependence

$$\frac{dy}{dt} = y(t) + t \qquad \frac{dy}{dt} = \sin(t) + 3 \cdot \sin(y(t)).$$

Autonomous 1st order equations can be solved by direct integration. The result is a "1-parameter family" of solutions w/ an arbitrary constant.

Ex 2.4 (Population Growth Equation)

y = Population

growth ∝ 3 (birthrate) × population

$$\frac{dy}{dt} = 3y(t)$$

$\downarrow$ pretend y is a independent variable

$$\frac{dy}{dt} = 3y$$

$\downarrow$ pretend $\frac{dy}{dt}$ is a fraction

$$\frac{1}{3y}\, dy = dt$$

$\downarrow$ Integrate both sides

$$\int \frac{1}{3y}\, dy = \int dt$$

$\downarrow$ evaluate

$$\frac{1}{3}\ln(|3y|) + C_2 = t + C_1$$

$\downarrow$ simplify $C_3 = 3C_1 - 3C_2$ is also an arbitrary constant.

$$\ln(|3y|) = 3t + C_3$$

$$\Leftrightarrow \quad 3y = \pm C_4 e^{3t} \qquad C_4 = e^{C_3}$$

$$\downarrow \qquad C_5 = \frac{\pm C_4}{3}$$

$$\boxed{y = C_5 e^{3t}}$$

Ex 2.5 (Drag Equation)

$$\frac{dy}{dt} = y(t)^2$$

$$\frac{dy}{dt} = y^2$$

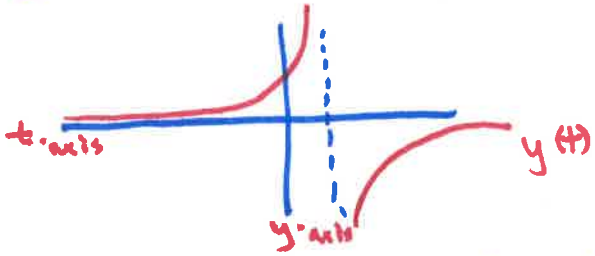
$$\int \frac{dy}{y^2} = \int dt$$

$$-\frac{1}{y} = t + C \qquad (C = -C_1 + C_2)$$

$$y(t) = -\frac{1}{t+C}$$

Important Remark 2.6: Solutions to some ODEs can "blow up" i.e. diverge in finite time.

Consider $C = -1$ in the above example



⚠

⇒ It is not true in general that the solutions of ODEs exist for all time.

2.ii Separable Equations: We could also do same as above w/ non-obvious integral on the right.

Def 2.7: A first-order ODE is said to be separable if it has the form

$$\frac{dy}{dt} = a(t) f(y(t)).$$

Note $\{\text{autonomous ODEs}\} \subseteq \{\text{Separable ODEs}\}$ w/ $a(t) = \text{const}.$

Ex 2.8 : Solve $\frac{dy}{dt} = 3t^2(y^2-1)$

$$\frac{dy}{y^2-1} = 3t^2\,dt$$

$$\int \frac{1}{y^2-1}\,dy = \int 3t^2\,dt$$

$$\frac{1}{2}\ln\left|\frac{y-1}{y+1}\right| = t^3 + C$$

$$\frac{y-1}{y+1} = De^{2t^3}$$

$$y(t) = \frac{1+De^{2t^3}}{1-De^{2t^3}}$$

Recall partial fraction decomposition

$$\frac{1}{y^2-1} = \frac{1}{2}\left(\frac{1}{y-1} - \frac{1}{y+1}\right)$$

$$\int \frac{1}{y^2-1} = \frac{1}{2}\left[\ln|y-1| - \ln|y+1|\right]$$

$$= \frac{1}{2}\ln\left|\frac{y-1}{y+1}\right|$$

Remark 2.9 : It is very easy to write down an ODE resulting in an integral with no anti-derivative in terms of standard functions.

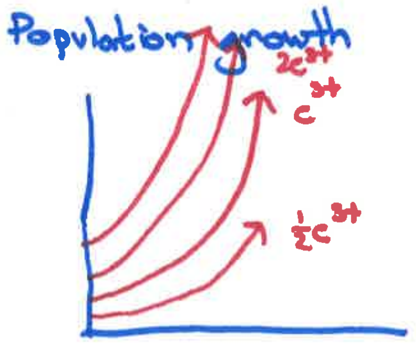
$$\frac{dy}{dt} = e^{-t^2}\cdot y$$

$$\ln|y| = \int e^{-t^2}\,dt \rightarrow$$

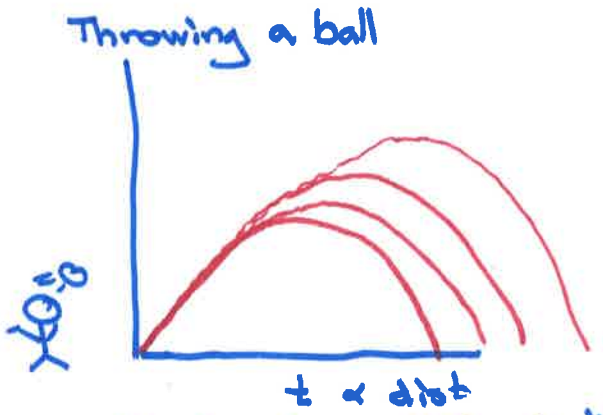
?? No anti-derivative in terms of $x, x^2, \ldots$ $\sin(x), \cos(x)$ e^x, $\log x$ . .

Upsetting fact of life 2.10 : not all ODEs have closed form solutions in terms of functions we have names for. The solutions still exist, but must be understood qualitatively or numerically.

2.iii) Initial-Value Problems :



All are valid solutions, for any D, they just have different populations at $t=0$.



Here depend on initial velocity.

Def 2.11: The initial value problem at time $t = t_0$ is to solve the coupled system of equations

$$\begin{cases} \frac{dy}{dt} = f(y(t),t) & \leftarrow \text{equality of functions } \forall t \\ y(t_0) = y_0 & \leftarrow \text{equality of numbers} \in \mathbb{R}. \end{cases}$$

A solution of the initial value problem is one defined for $I = (t_0 - a, t_0 + b)$.

$\Rightarrow$ We solve an IVP by first solving general form w/ arbitrary constant C, then plugging in to find C.

Ex 2.12: Find the population at time t if it solves $P(0) = P_0$.

$$\begin{cases} \frac{dP}{dt} = 5P(t) \\ \quad\text{or } \lambda \\ P(0) = P_0 \end{cases}$$

As before $P(t) = De^{5t}$

Plug in $\Rightarrow$ $P_0 = P(0) = De^{5 \cdot 0} = D$, so $D = P_0 = P(0)$.

$\boxed{P(t) = P_0 e^{5t}}$ More generally. $P(t) = P(0)e^{5t}$.
$c^{\lambda t}$

Ex 2.13: See Ex 2.3.7 in book, and
Ex 2.3.8

Ex 2.14 (Oil drop) Velocity of falling oil drop mass m

$$m\frac{dv}{dt} = mg - K \cdot v(t)$$

$\nwarrow$ drag constant

Initial velocity $v(0) = v_0$. Not Separable w/ solution given in a few lectures

$$v(t) = \left(v_0 - \frac{mg}{K}\right)e^{-\left(\frac{K}{m}\right)t} + \frac{mg}{K}$$

so $\lim_{t\to\infty} v(t) = \frac{mg}{K}$. Terminal Velocity, used by Millikan to find electron change.

$\boxed{\textbf{Theorem 2.15}}$ Uniqueness of I.V.P. Given an ODE

$$\frac{dy}{dt} = f(y(t),t)$$

and a point $(t_0, y_0) \in \mathbb{R}^2$ there is a unique solution of IVP w/ graph through (t_0, y_0). ie. there is exactly 1 solution.

Lecture 3 | First-Order ODEs : dynamical perspective.

When we solve an IVP

$$\begin{cases} \frac{dy}{dt} = f(y(t),t) \\ y(0) = y_0 \end{cases}$$

we are asking, given knowledge of the system at time t_0, what are dynamical propertio
(i.e. how does the system behave or change moving forward in time?

Recall Upsetting Fact 2.10 : we cannot solve all ODEs explicitly. Fortunately, properties we are interested in don't need this!

Ex 3.1 : In oil drop experiment, what we ultimately wanted was the terminal velocity $\lim_{t\to\infty} v(t)$. Don't need to know $v(t_1)$ for small t_1.

3.i) Two Important ODEs

A simplistic model of heat : the amount of heat transfer between objects in proportional to the difference in temperature

Obviously temp depends on space too, we return to the heat eq (more sophisticated) later

Law 3.2 : (Newton's Law of Cooling) The temperature of an object $y(t)$ in ambient temperature $T(t)$ has dynamics modeled by

$$\frac{dy}{dt} = -K\left(y(t) - T(t)\right)$$

where K = constant depending on material (units = $1/\text{time}$).

Qualitative Question 3.3 : i) How long after taking my tea out of the microwave can I drink it w/o burning my tongue if my room is 70°F?
(ii) If global ambient air temp rises 2°C, when will polar ice caps be half-melted?

Ex 3.4 : If $T(t) = T_0$ is ind. of time,

$$\frac{1}{y-T_0}\frac{dy}{dt} = -K \implies y(t) = T_0 + De^{-Kt}.$$

$$\ln|y-T_0| = \cancel{e}^{-Kt} + C$$

$\boxed{3.2}$

Ex 3.5: Reversing the logic. Suppose in an $T_0 = 80°F$ room we measure

$y(0) = 160°F$

$y(20) = 145°F$. What is the constant k?

1) $y(t) = T_0 + De^{-kt}$ solve for D using IVP

$y(0) = T_0 + De^0 = T_0 + D \Rightarrow D = y(0) - T$

2) $y(t) = T_0 + (y(0) - T_0)e^{-kt}$

$y(20) - T_0 = (y(0) - T_0)\, e^{-k(20)}$

$65 = 80e^{-20k} \Rightarrow k = -\frac{1}{20}\ln\left(\frac{65}{80}\right) = 0.01038/\text{min}$

Idea 3.6: Population growth is not exponential forever, but is resource limited. A more sophisticated/realistic model is the logistic ODE

$$\frac{dy}{dt} = Ry(t)\left(1 - \frac{y(t)}{K}\right)$$

R: growth rate

K: Carry capacity.

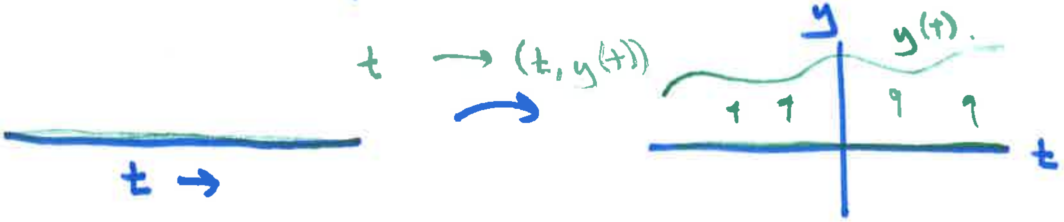
Intuition, if $y(t) << K$, then $\frac{dy}{dt} \approx Ry(t)$ is unconstrained but

$y \approx K$, then $\frac{dy}{dt} \approx Ry(t) \cdot 0 = 0$ is const.

Can we get qualitative information w/o actually solving?

3.ii) Slope fields

Recall the graph is a way of picking up the t-axis and putting it down in the 2d plane

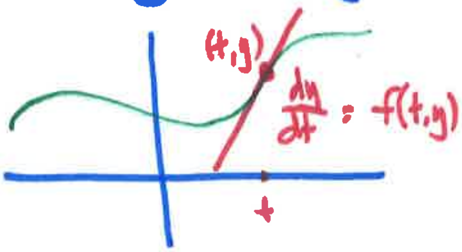


$\boxed{3.3}$

If $y(t)$ solves a particular ODE

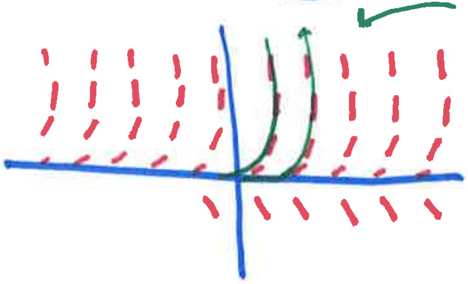
$$\frac{dy}{dt} = f(y(t),t)$$

then its graph at (t,y) is tangent to $f(y(t),t)$



<u>Def 3.7</u> : The <u>slope field</u> of an ODE $\frac{dy}{dt} = f(y(t),t)$ is the plot of the line segments of slope $f(y,t)$ at each (t,y).

<u>Ex 3.8</u> : $\frac{dy}{dt} = 3y$

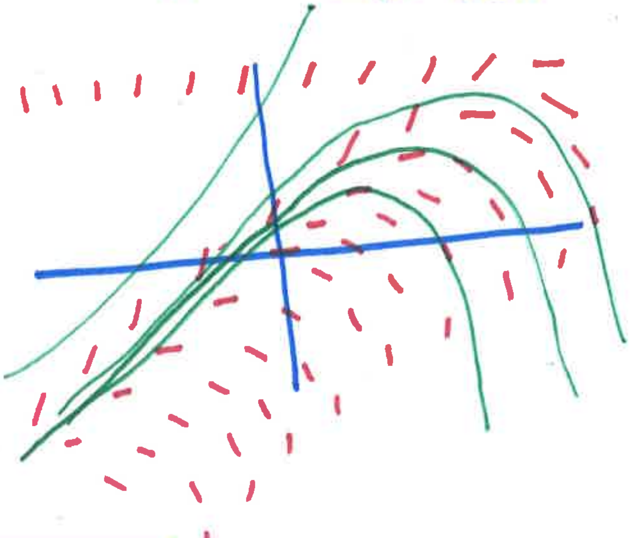


Solution curves (compare shape w/ known sol De^{3t}).

<u>Equivalence 3.9</u> : The solution of the initial value problem at (t_0, y_0) is the (unique) curve through (t_0, y_0) everywhere tangent to the slope field.

<u>Ex 3.10</u> : Consider $\frac{dy}{dt} = y - t$

a) Draw the slope field

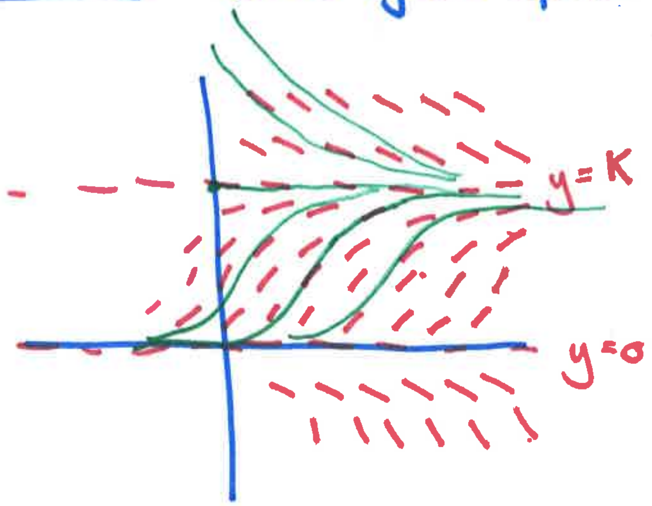


b) Find the slope of solution curve at

$(-2,0)$	$\rightarrow 2$
$(0,-3)$	$\rightarrow -3-0 = -3$
$(2,-1)$	$\rightarrow -1-2 = -3$
(a,b)	$\rightarrow b-a$

Ex 3.11: If $\frac{dy}{dt} = f(y)$ is autonomous, slope field + solutions are horizontal translation invariant.

Ex 3.12 (Back to logistic equation) $\frac{dy}{dt} = Ry\left(1 - \frac{y}{K}\right)$ ↖ quadratic in y



· Population over K decreases exponentially
· under K grows and approaches K asymptotically.

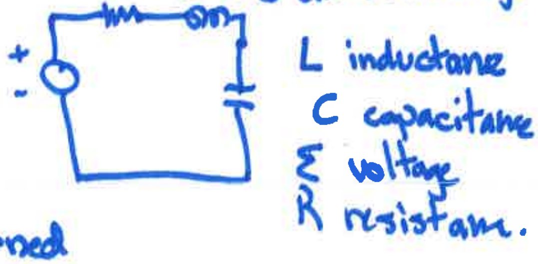
3.iii) Poincaré's Principle

Consider:

$$m y''(t) + c y'(t) + k y(t) = \underbrace{F(t)}_{\text{driving force}}$$

$$m l \theta''(t) + b l \theta'(t) + m g \theta(t) = F(t)$$

$$L I''(t) + R I'(t) + \frac{1}{C} I(t) = \mathcal{E}(t)$$



Observe these are all the same ODE just with constants interpreted w/ different meaning!

Principle 3.13: If two physical systems are governed by the same ODE, physical + mathematical insights from one system can be applied to others!

e.g. it's physically obvious pendulum oscillates, but much harder to be able to look at the circuit and intuit that it also oscillates!

Ex 3.14 "Scientists create baby wormhole in lab as sci-fi moves closer to fact" - CNN, Reuters, Dec 1 2022.

No! They did not create a wormhole, they purposely designed a system governed by an analogous ODE!

Lecture 4 Stationary Values, Stability, and Phase Lines

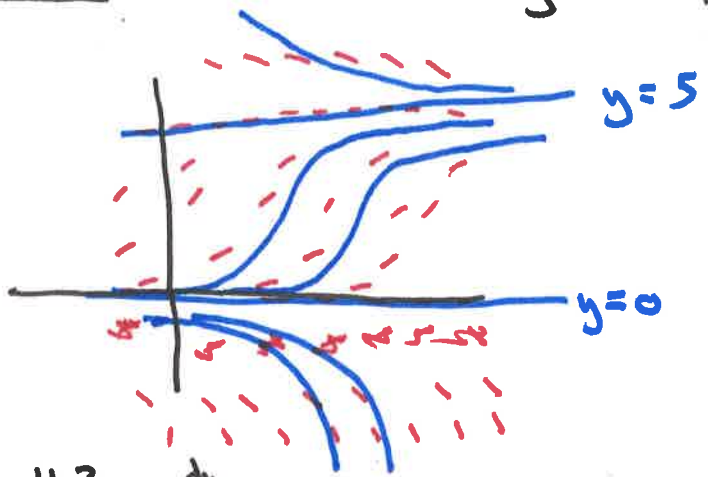
Recall that logistic eq, Newton's law, oil drops all model systems that approach a physical equilibrium as t → ∞.

4.i) Stationary Values

Def 4.1: A stationary $\overset{\text{point/value/}}{\text{solution}}$ of an (autonomous) $\overset{\frac{dy}{dt} = f(y)}{\text{ODE}}$ is a solution

$y(t) = c$ for $c \in \mathbb{R}$ constant, where $f(c) = 0$.

Ex 4.2: Consider the logistic equation $\frac{dy}{dt} = Ry\left(1 - \frac{y}{K}\right)$ eg $y\left(1 - \frac{y}{5}\right)$



$0 = f(y) = y\left(1 - \frac{y}{5}\right)$

$\Rightarrow y = 0$ or 5 $(0, K)$ more generally

are stationary values.

Ex 4.3 $\frac{dy}{dt} = y^2 - 1$ has stationary values $y^2 - 1 = 0$

$(y+1)(y-1) = 0$

$y = \pm 1$.

Ex 4.4: $\frac{dy}{dt} = y^2 + 1$ has no stationary values because $y^2 + 1 = 0$ has no solutions in $\mathbb{R}$.

Observe above that solution curves converge to $y = 5$ as $t \to \infty$ but diverge from $y = 0$.

Def 4.5: A stationary value $y(t) = c$ of $\frac{dy}{dt} = f(y(t))$ is called

i) stable

ii) unstable

iii) semistable

$$* \begin{cases} \frac{dy}{dt} = f(y(t)) \\ y(0) = y_0 \end{cases}$$

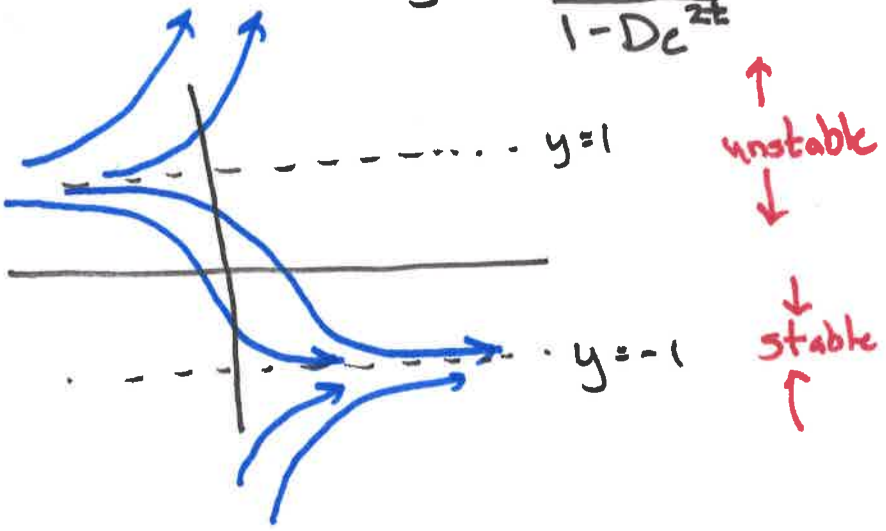
if the IVP * has solutions which i) converge to c for $y_0 \in (c-\delta, c+\delta)$

ii) $y(t)$ always departs $(c-\delta, c+\delta)$ as $t \to \infty$

iii) One of each for $y > c$, $y < c$.

Ex 4.6: Recall that $\frac{dy}{dt} = y^2 - 1$ has solutions

$$y(t) = \frac{1 + De^{2t}}{1 - De^{2t}}$$



Ex 4.7: Stable is a very useful property for measurement: near a stable equilibrium measurement error does not affect prediction!

Consider

If $\theta(0) = 0$ $\theta(t) = 0$ for all t

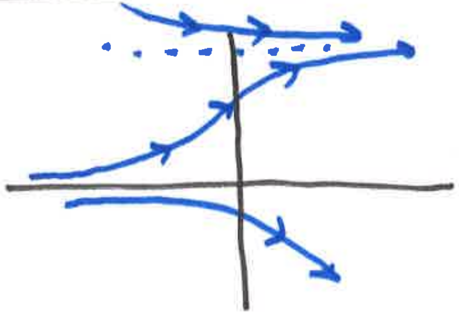
$\theta(0) = 0.0000001$ sphere falls

In ex 4.6, $y(0) = 1$ $y(t) = 1$ all t

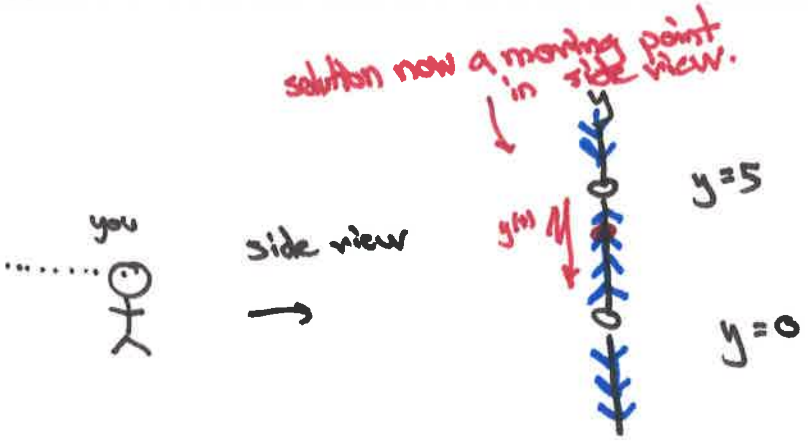
but $y(0) = 1.000001$ $y(t) \to \infty$.

To predict accurately at an unstable equilibrium would require infinite precision!

4.ii) Phase lines

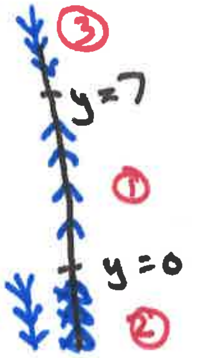


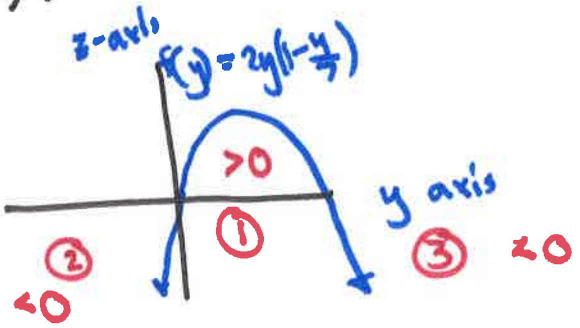
2-dimensional solution curves



Ex 4.8 : Consider a particle at height y governed by

$$\frac{dy}{dt} = 2y\left(1 - \frac{y}{7}\right)$$





In region ① particle is trapped, and increases towards $y=7$ b/c $\frac{dy}{dt} > 0$ there

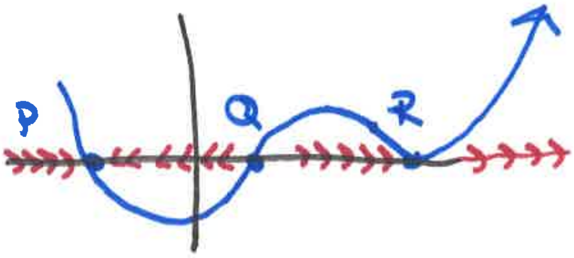
②③ particle escapes for ② but decreases to $y=7$ for ③ b/c $\frac{dy}{dt} < 0$ there

⇒ Qualitative dynamics can be concluded without solving the equation! Solving needed for quantitative information.

Def 4.9 : For an autonomous ODE $\frac{dy}{dt} = f(y)$, the plot of directed vertical motion for initial values ≠ stationary is called the <u>phase diagram</u> or phase line.

Note on the phase diagram, each 'region' ie segment must converge to a stationary value as $t \to \infty$ <u>or</u> diverge to $\pm\infty$.

Ex 4.10 : Consider a polynomial $f(y) = P(y)$



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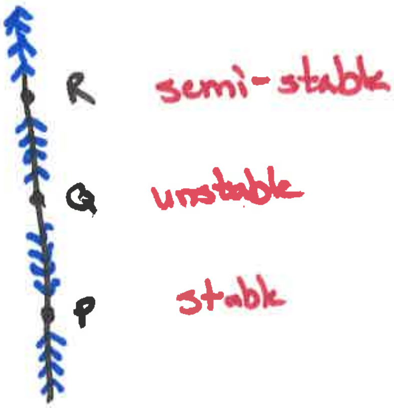


Just looking at the sign of the graph tells us the direction of motion.

$\boxed{4.4}$

Thm 4.11 : If $\frac{dy}{dt} = f(y)$ is an autonomous 1st order equation, and $y(t) = c$ is stationary, then

i) c is stable if $f(y)$ crosses from + to − at c

ii) c is unstable if − to + at c

iii) c is semistable if + to + or − to −

Ex 4.12 : Consider $\frac{dy}{dt} = y^2$

1) Find the stationary points and classify them, draw phase diagram

$y^2 = 0$

$y = 0$ stationary point.

$y > 0$ for $y \neq 0$, $y = 0$ is semi-stable.

2) Note semi-stable has both $f(c) = 0$
$f'(c) = 0$.

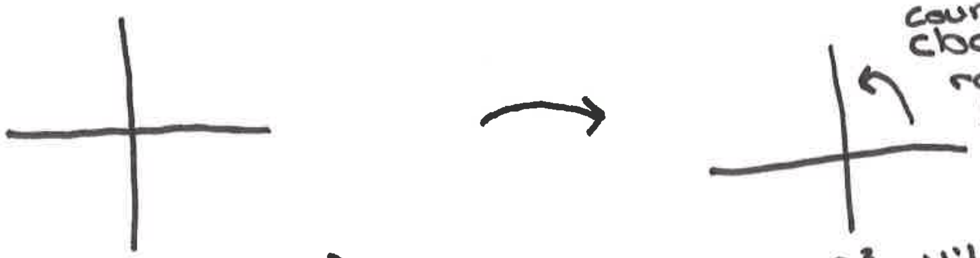
To distinguish  from  we use second-derivative test

Ex 4.13 : $f(y) > 0$ or $f(y) < 0$ we know increasing or decreasing. Can also get concave up or down by calculating

$$y''(t) = \frac{d}{dt} y'(t) = \frac{d}{dt}\frac{dy}{dt} = \frac{d}{dt} f(y(t)) = f'(y) \cdot y'(t)$$
$$= f'(y) \cdot f(y)$$

For $\frac{dy}{dt} = y^2$ $\quad f(y) = y^2$ $\quad f'(y) = 2y$ $\qquad f'(y) \cdot f(y) = 2y^3$ > 0 for $y > 0$, < 0 for $y < 0$.

Lecture 5 Complex Numbers



counter-clockwise rotation by 90° $\begin{pmatrix}1\\0\end{pmatrix}\to\begin{pmatrix}0\\1\end{pmatrix}$ $\begin{pmatrix}0\\1\end{pmatrix}\to\begin{pmatrix}-1\\0\end{pmatrix}$ $\to\begin{pmatrix}0&-1\\1&0\end{pmatrix}$

$\mathbb{R}^2=\{\begin{pmatrix}x\\y\end{pmatrix} \mid x,y\in\mathbb{R}\}$ vector space (scaling, +)

Gives $\mathbb{R}^2$ additional information about angles + rigid geometry

Ex 5.0: Schrödinger's equation $i\hbar\frac{dy}{dt}=H(y(t))$ says time change is rotation, conveniently captured by complex numbers

5.i Complex Numbers

Def 5.1: The complex numbers are the data

$$\mathbb{C}=(\mathbb{R}^2, i)$$

where $i=\begin{pmatrix}0&-1\\1&0\end{pmatrix}:\mathbb{R}^2\to\mathbb{R}^2$ is rotation. The basis is denoted $1=\begin{pmatrix}1\\0\end{pmatrix}$ $i=i\begin{pmatrix}1\\0\end{pmatrix}=\begin{pmatrix}0\\1\end{pmatrix}$ $=i\cdot 1$.

$$=\{z=x+iy \mid x,y\in\mathbb{R}^2\}.$$

Fact 5.2: $i^2=-1$. This is geometrically obvious, it's a 180° rotation.

$$\begin{pmatrix}0&-1\\1&0\end{pmatrix}\begin{pmatrix}0&-1\\1&0\end{pmatrix}=\begin{pmatrix}-1&\\&-1\end{pmatrix}=-\mathrm{Id}.$$

⇒ This gives a way to multiply complex numbers

Ex 5.3: Complex numbers add like basis vectors

$$z_1+z_2=(x_1+iy_1)+(x_2+iy_2)$$
$$=(x_1+x_2)+i(y_1+y_2)$$

$$(5+3i)+(7-4i)=12-i.$$
$$3 \quad -(6-8i)=-3+8i$$

Ex 4: Complex numbers multiply by distributing + using $i^2=1$

$$(5+2i)(3+4i)=5\cdot3+2i\cdot3+5\cdot(+4i)+2i\cdot(+4i)$$
$$=15+6i+20i+\underbrace{8i^2}_{=-8}$$
$$=7+26i$$

Ex 5.5: To divide by a complex number multiply by

$$\frac{x-iy}{x-iy} = \frac{\bar{z}}{\bar{z}} \qquad z = x+iy$$

To get real denominators

$$\frac{1}{3+5i} = \frac{1}{3+5i}\cdot\frac{3-5i}{3-5i} = \frac{3-5i}{(3+5i)(3-5i)} = \frac{3-5i}{34}$$
$$= \frac{3}{34} - \frac{5}{34}i$$

Def 5.6: For $z = x+iy$, $\bar{z} = x-iy$ is called the complex conjugate. Note that is satisfies

$$z\bar{z} = |x|^2 + |y|^2.$$

Def 5.7: For $z = x+iy \in \mathbb{C}$,

$x \in \mathbb{R}$ is called the Real Part, denoted $Re(z)$

$y \in \mathbb{R}$ is called the imaginary part, denoted $Im(z)$.

Observe that

$$Re(z) = \frac{1}{2}(z+\bar{z}) \qquad Im(z) = \frac{1}{2i}(z-\bar{z})$$

⚠ Notice that the imaginary part $Im(z) \in \mathbb{R}$ is a real number. It is the coefficient of i in the basis, not $iy \in \mathbb{C}$.

The sets $\mathbb{R} = \{x + 0i \mid x \in \mathbb{R}\} \subseteq \mathbb{C}$
$i\mathbb{R} = \{0 + iy \mid y \in \mathbb{R}\} \subseteq \mathbb{C}$ are called the real / Imaginary axis.

Proposition 5.8: The following relations hold

- $\overline{z+w} = \bar{z} + \bar{w}$ $\qquad \overline{(x+iy)} + \overline{(\alpha+i\beta)} = x-iy+\alpha-i\beta = (x+\alpha)-(y+\beta)i$ ✓
- $\overline{z\cdot w} = \bar{z}\cdot\bar{w}$ $\qquad$ Similar
- $\overline{\left(\frac{z}{w}\right)} = \frac{\bar{z}}{\bar{w}}$
- $\overline{(\bar{z})} = z$ $\qquad \overline{\overline{x+iy}} = \overline{x-iy} = x+iy$ ✓

Prop 5.9: Complex numbers have $\sqrt{\ }$ defined for ~~No~~ $c \in \mathbb{R}$ (not $c>0$).

$$(\pm i\sqrt{c})^2 = -1\cdot c = -c \quad \text{so} \quad \sqrt{-c} = \pm i\sqrt{c} \text{ for } c>0.$$

5.ii) Algebra of Complex numbers

Recall that a real polynomial $P(x) = a_n x^n + a_{n-1} x^{n-1} + \dots + a_1 x + a_0$ must have $\leq$ degree $= n$ roots.

Ex 5.10: $P(x) = x^2+1$ no real roots, cannot be factored over $\mathbb{R}$

$= (x+i)(x-i)$ roots exist over $\mathbb{C}$, factorable.

We can also allow $a_n \in \Rightarrow \alpha_n \in \mathbb{C}$ $x \Rightarrow z \in \mathbb{C}$.

$$P(z) = \alpha_n z^n + \alpha_{n-1} z^{n-1} + \dots + \alpha_1 z + \alpha_0$$

Thm 5.11: (Fundamental Theorem of Algebra) Any polynomial $P(z)$ of degree n w/ coefficients in $\mathbb{C}$ $(\alpha_n \neq 0)$ has precisely n real roots $z_1, \dots, z_n \in \mathbb{C}$. Thus it factors completely as

$$P(z) = (z-z_1)(z-z_2) \dots (z-z_n).$$

Ex 5.11: Qudratic Formula applies for $n=2$

az^2+bz+c has $z_1, z_2 = \dfrac{-b \pm \sqrt{b^2-4ac}}{2a}$ ✓ maybe <0.

5.iii) The Geometry of Complex numbers

Recall $\mathbb{R}^2$ may be given polar coordinates (r, θ) so $(x,y) = (r\cos\theta, r\sin\theta)$.

$\Rightarrow z \in \mathbb{C}$ may be written in polar coordinates as
$$z = x+iy = r(\cos\theta + i\sin\theta)$$

Def 5.12: For $z \in \mathbb{C}$, the complex exponential is defined by

$$e^z := \sum_{n=0}^{\infty} \frac{z^n}{n!} = 1 + z + \frac{z^2}{2} + \frac{z^3}{3!} + \dots$$ ✓ converges for all z by e.g. ratio test

Thm 5.13 (Euler's formula)

1) The identity $$e^{i\theta} = \cos\theta + i\sin\theta$$ holds for $\theta \in \mathbb{R}$

2) Sin, cos make sense for complex numbers via
$$\cos z = \tfrac{1}{2}(e^{iz} + e^{-iz}) \qquad \sin z = \tfrac{1}{2i}(e^{iz} - e^{-iz}).$$

Proof : Recall Taylor series

$$\cos\theta = 1 - \frac{\theta^2}{2} + \frac{\theta^4}{4!} + \dots$$

$$\sin\theta = \theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} + \dots$$

$$e^{i\theta} = 1 + i\theta + \frac{(i\theta)^2}{2!} + \frac{(i\theta)^3}{3!} + \dots$$

$$= \underbrace{\left(1 + (i)^2\frac{\theta^2}{2!} + (i)^4\frac{\theta^4}{4!} \dots\right)}_{= \cos\theta} + i\underbrace{\left(\theta + (i)^2\frac{\theta^3}{3!} + (i^4)\frac{\theta^5}{5!} + \dots\right)}_{i\sin\theta.}$$

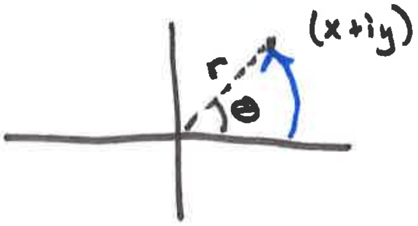
$$e^{i\theta} + e^{-i\theta} = \cos\theta + i\sin\theta + \cos(-\theta) + i\sin(-\theta) = 2\cos\theta. \quad \checkmark.$$

$$e^{i\theta} - e^{i\theta} = 2i\sin\theta$$

$\square$

Def 5.13 : The polar expression for a complex number z is

$$\boxed{z = re^{i\theta}}$$



where

Called the modulus or abs. value of z

$$|z| = r = \sqrt{x^2+y^2} = \sqrt{z\bar{z}}$$

$$\tan\theta = \frac{y}{x}$$

Ex 5.14 : $i = e^{i\pi/2}$ $\quad -1 = e^{i\pi}$

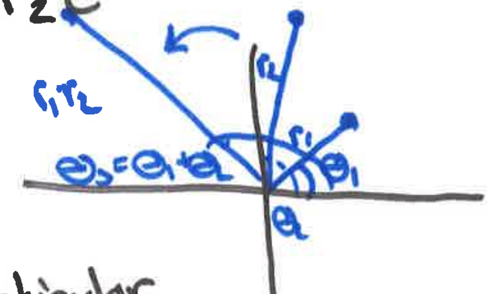
$2i = 2e^{i\pi/2}$ $\quad 1 = e^{2i\pi} = e^{4\pi i} = \dots$

Prop 5.15 : The complex exponential satisfies

- $e^z \cdot e^w = e^{z+w}$
- $e^{x+iy} = e^x e^{iy}$

Prop 5.16 : Multiplication of complex numbers in polar form obeys

$$(r_1e^{i\theta_1})(r_2e^{i\theta_2}) = r_1r_2e^{i(\theta_1+\theta_2)}$$



Ex 5.17 : $\sqrt{re^{i\theta}} = \sqrt{r}e^{i\theta/2}$ in particular

$$\sqrt{i} = \sqrt{1}e^{i\pi/4} = \frac{1}{\sqrt{2}} + \frac{i}{\sqrt{2}}$$

Lecture 6 Second-Order Linear ODEs

Newton's 2nd law F = ma can be phrased as a second-order ODE

$$m\frac{d^2y}{dt^2} = F\left(t, y, \frac{dy}{dt}\right)$$

acceleration

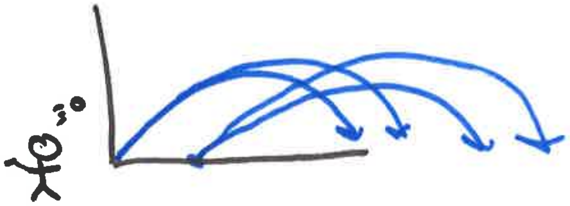
Force depends on current time, position, velocity

⇒ Second-order equations are ubiquitous in physics.

Applications 6.1 : Second-order systems arise in the following places

1) Planets / celestial mechanics
2) Electrostatics + circuits
3) Vibrations + oscillations
4) any mechanical system you can think of

Ex 6.2 : Consider throwing a ball



To know the trajectory we must know two numbers, the initial position and the initial value

⇒ Second-order IVP has two unknown constants C_1, C_2 depending on $y(0)$, $y'(0)$.

6.i) Exponential Ansatz

It is sometimes hard to find solutions to ODEs, but it is generally easy to verify if something is a solution (just plug in).

Ansatz 6.3 : For (linear) second-order equations, it is often a good guess to suppose the solution(s) have the form

$$y(t) = e^{\lambda t} \quad \text{for } \lambda \in \mathbb{C}.$$

Ex 6.4 : $\frac{d^2y}{dt^2} - y = 0$ If $y = e^{\lambda t}$ $y' = \lambda e^{\lambda t}$ $y'' = \lambda^2 e^{\lambda t}$.

$$\lambda^2 e^{\lambda t} - 1 e^{\lambda t} = (\lambda^2 - 1)e^{\lambda t} = 0 \quad \Rightarrow (\lambda^2 - 1) = 0$$

$$\lambda = \pm 1.$$

e^x is never 0.

$\Rightarrow$ In fact same for $De^{\lambda t}$ so

$$y(t) = D_1 e^{t} \quad , D_2 e^{-t}$$

are both solutions.

Observe that the ansatz reduced solving an ODE to solving a polynomial equation.

6.ii) Linear Equations

Def 6.5: A second-order ODE is said to be linear if it is a linear function of $\frac{d^2y}{dt^2} = y''$ $\frac{dy}{dt} = y$ and y, ie it has the form

$$a(t)y'' + b(t)y' + c(t)y = 0.$$

It has constant coefficients if $a,b,c \in \mathbb{R}$ are t-independent.

Ex 6.6:

$\sin(t)y'' + t^2 y = 0$ is linear (y, y'' appear linearly)

$\sin(\dot{y})t + y^2 t = 0$ is not ($\sin(y'')$ is non-linear in y).

$a(t)(y'')^2 + c(t)y = 0$ is not

Thm 6.7 (Principle of Superposition) If y_1, y_2 are both solutions of a linear second-order equation, then

$$y(t) = C_1 y_1(t) + C_2 y_2(t)$$

is also a solution for any $C_1, C_2 \in \mathbb{R}$.

Proof: If y_1, y_2 solve then

$$C_1 \cdot \quad a(t)y_1'' + b(t)y_1' + c(t)y_1 = 0$$

$$C_2 \cdot \quad a(t)y_2'' + b(t)y_2' + c(t)y_2 = 0$$

$$\xrightarrow{\hspace{6em}}$$

$$a(t)(C_1y_1'' + C_2y_2'') + b(t)(C_1y_1' + C_2y_2') + c(t)(C_1y_1 + C_2y_2) = 0. \qquad \square$$

Def 6.8: For two functions y_1, y_2 not scalar multiples

$$C_1y_1(t) + C_2y_2(t)$$

is called a <u>linear combination</u>.

Thm 6.9: If $a(t)y'' + b(t)y' + c(t)y = 0$ is a linear second-order equation, then there are precisely two linearly independent solutions $y_1(t), y_2(t)$ such that any solution is a linear combination

$$y(t) = C_1 y_1(t) + C_2 y_2(t).$$

In particular, the IVP

$$\begin{cases} ay'' + by' + cy = 0 \\ y(0) = y_0 \\ y'(0) = v_0 \end{cases}$$

has a single unique solution (same at $t = t_0$).

Observe if y_1 y_2 are found by ansatz, these are necessarily all the solutions

6.iii) Constant Coefficients and Characteristic Polynomials

If $a, b, c \in \mathbb{R}$ assume $y(t) = e^{\lambda t}$ solves $ay'' + by' + cy = 0$

then

$$(a\lambda^2 + b\lambda + c)e^{\lambda t} = 0$$

$$a\lambda^2 + b\lambda + c = 0$$

$$\lambda = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Def 6.10: This is called the characteristic polynomial of the equation.

Thm 6.11: A constant coefficient equation has "basic solution"

i) $b^2 - 4ac > 0$ two distinct real roots λ_1, λ_2

$$y_1(t) = e^{\lambda_1 t} \quad y_2(t) = e^{\lambda_2 t}$$

ii) $b^2 - 4ac < 0$ Imaginary parts negatives, ie two complex roots

$$\lambda_1 = r_0 + i w_0 \qquad \Rightarrow \qquad y_1(t) = e^{r_0 t}\cos(w_0 t)$$
$$\bar{\lambda}_1 = r_0 - i w_0 \qquad\qquad y_2(t) = e^{r_0 t}\sin(w_0 t)$$

Re, Im of $e^{\lambda_1 t}$ $e^{\bar{\lambda}_1 t}$

iii) $b^2 - 4ac = 0$ single real root λ_1

$$y_1(t) = e^{\lambda_1 t}$$
$$y_2(t) = te^{\lambda_1 t}.$$

Ex 6.12: Consider a mass-spring system $my'' + by' + ky = 0$

$\leftarrow$

ky elastic force

$-by'$ friction

Take values

$$y'' + 2y' + 2y = 0.$$

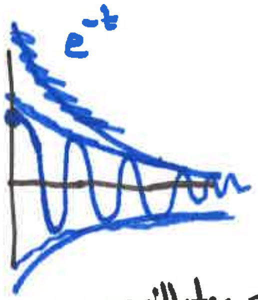
a) Find the general sol.

$\Rightarrow \lambda^2 + 2\lambda + 2 = 0$ characteristic polynomial.

Therefore, $\lambda = \dfrac{-2 \pm \sqrt{4 - 4 \cdot 2 \cdot 1}}{2} = -1 \pm i$

So the general solution is

$$y(t) = C_1 e^{-t}\cos(t) + C_2 e^{-t}\sin(t).$$



(mass oscillates then slows and comes to rest.)

b) If $y(0) = 3$, $y'(0) = 0$. find the solution of the IVP

$y(0) = C_1 \cdot 1 + C_2 \cdot 0 \quad \Rightarrow C_1 = 3$

$y'(t) = -C_1 e^{-t}\sin(t) + -C_1 e^{-t} \cdot \cos(t) + C_2 e^{-t}\cos(t) - C_2 e^{-t}\sin(t)$

$y'(0) = -C_1 - C_2 = 0 \quad \Rightarrow C_2 = -3.$

$$y(t) = 3e^{-t}\cos(t) - 3e^{-t}\sin(t).$$

Prop 6.13: (time-translation principle) For a (homogeneous) const coeff. second-order IVP at $t_0 = 0$, w/ solution $Y(t)$, then $Y(t - t_0)$ is a solution w/ same IVP at $t_0 = t_0$.

Ex 6.14: Consider $\begin{cases} y'' - 6y' + 9y = 0. \\ y(-4) = 1 \\ y'(-4) = 2 \end{cases}$

$\lambda^2 - 6\lambda + 9 = 0 \quad \Rightarrow (\lambda - 3)^2 = 0 \quad \Rightarrow \lambda = 3$ (repeated)

$$y(t) = C_1 e^{3t} + C_2 t e^{3t}$$

Solve at $t = 0$. $\begin{cases} y(0) = 1 \\ y'(0) = 2 \end{cases}$ $\quad \Rightarrow C_1 = 1$

$2 = 3C_1 e^{3t} + C_2 e^{3t} + 3C_2 e^{3t} \cdot t$

$2 = 3 + C_2 \quad C_2 = -1$

$$\boxed{y = e^{3(t+4)} + (-1)(t+4)e^{3(t+4)}}$$

Lecture 7 Linear Systems of ODEs

Many physical systems have more than one quantity to model, and these quantities interact!

7.i) Systems of ODEs as matrices

Ex 7.1 (Predator-Prey systems)

$y_1(t) = W(t)$ populations of wolves

$y_2(t) = M(t)$ moose

$$\frac{dW}{dt} = R_1 W(t) + \boxed{P_1 M(t)}$$

$$\frac{dM}{dt} = \boxed{-P_2 W(t)} + R_2 M(t)$$

→ Predation rate P_i give interaction/coupling terms.

→ birthrates R_i depend on self-populations

Let $\vec{y}(t) = \begin{bmatrix} y_1(t) \\ y_2(t) \end{bmatrix}$

the system can be written as a matrix $\frac{d}{dt}[\vec{y}] = A[\vec{y}]$ $A = \begin{pmatrix} R_1 & P_1 \\ -P_2 & R_2 \end{pmatrix}$

Ex 7.2 : (F=ma, 3d) $m\begin{bmatrix} x'' \\ y'' \\ z'' \end{bmatrix} = F\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}, \begin{bmatrix} x' \\ y' \\ z' \end{bmatrix}\right)$ particle moving In 3 dimensions.

Def 7.3 : A (coupled) linear system of const. coefficient first-order ODEs is a matrix equation

$$\frac{d}{dt}\begin{bmatrix} y_1 \\ \vdots \\ y_n \end{bmatrix} = A\begin{bmatrix} y_1 \\ \vdots \\ y_n \end{bmatrix} + \begin{bmatrix} f_1(t) \\ \vdots \\ f_n(t) \end{bmatrix}$$

for functions $f_1 \ldots f_n(t)$, $y_1 \ldots y_n(t)$. It is homogeneous if $f_i = 0$.

7.ii) Solving First-order Systems with Eigenvectors

Ex 7.4: Consider

$$\frac{d}{dt}\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} \lambda_1 & \\ & \lambda_2 \end{bmatrix}\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} \Rightarrow \begin{aligned} \frac{dy_1}{dt} &= \lambda_1 y_1 \\ \frac{dy_2}{dt} &= \lambda_2 y_2 \end{aligned}$$

Diagonal system decouples into two separate eqns solvable by separable/autonomous methods of Ch.2.

Key Idea 7.5: General systems may be solved by a change of basis reducing them to diagonal systems.

Recall,

Def 7.6: If $A \in \mathrm{Mat}(\mathbb{R}^n, \mathbb{R}^n)$ is $n \times n$, then x is an eigenvector if

$$Ax = \lambda x$$

for some $\lambda \in \mathbb{R}$, called its eigenvalue.

If λ eigenvalue, $\det(A - \lambda \mathrm{Id}) = 0$ and x spans nullspace.

Ex 7.7: Find the eigenvectors + eigenvalues of

$$A = \begin{pmatrix} 6 & -20 \\ 1 & -3 \end{pmatrix}$$

1) (Eigenvalues) λ solve

$$\begin{aligned} 0 &= \det(A - \lambda \mathrm{Id}) \\ &= \det\begin{pmatrix} 6-\lambda & -20 \\ 1 & -3-\lambda \end{pmatrix} = (6-\lambda)(3-\lambda)+20 = \lambda^2 - 3\lambda + 2 \\ &\Rightarrow \lambda = 1, 2. \end{aligned}$$

2) (Eigenvectors) nullspace of $(A - 1 \cdot \mathrm{Id})\vec{x}_1 = 0$

$$\begin{pmatrix} 5 & -20 \\ 1 & -4 \end{pmatrix}(x) = 0 \quad \Rightarrow \text{row reduction}$$

$$\begin{pmatrix} 1 & -4 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}, \quad \vec{x} = \begin{pmatrix} 4 \\ 1 \end{pmatrix}$$

nullspace of $(A - 2\mathrm{Id})\vec{x}_2 = 0$

$$\begin{pmatrix} 4 & -20 \\ 1 & -5 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} \quad \Rightarrow \quad \vec{x}_2 = \begin{pmatrix} 5 \\ 1 \end{pmatrix}$$

Ex 7.8 (Shortcut for 2x2 matrices)

$$\begin{aligned} \det\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix} - \lambda \mathrm{Id}\right) &= (a-\lambda)(d-\lambda) - bc \\ &= \lambda^2 - (a+d)\lambda + (ad-bc) = \lambda^2 - \mathrm{Tr}(A)\lambda + \det(A). \end{aligned}$$

Def 7.9: $\det(A - \lambda \mathrm{Id})$ is called the characteristic polynomial of the matrix A.

Prop 7.10: If $U = \begin{pmatrix} \vec{x}_1, \ldots \vec{x}_n \end{pmatrix}$ is the change of basis matrix with eigenvector columns,

then
$$A = U \begin{pmatrix} \lambda_1 & & & \\ & \lambda_2 & & \\ & & \ddots & \\ & & & \lambda_n \end{pmatrix} U^{-1}$$

i.e. A is diagonal in the eigenbasis.

Thm 7.11 (Decoupling Theorem)

Let A be an $n\times n$ matrix w/ eigenbasis $v_1 \dots v_n$ w/ eigenvalues $\lambda_1 \dots \lambda_n \in \mathbb{R}$

Let $U = (v_1 \dots v_n)$ as before.

1) If $y(t) = \sum_{k=1}^{n} u_k(t) v_k$ is the function written in the eigenbasis, ie.

$\begin{bmatrix} u_1 \\ \vdots \\ u_n \end{bmatrix} = U^{-1} \left(\begin{bmatrix} y_1 \\ \vdots \\ y_n \end{bmatrix} \right)$ then u_k solve the decoupled diagonal system

$$\frac{d}{dt}\begin{bmatrix} u_1 \\ \vdots \\ u_n \end{bmatrix} = \underbrace{\begin{bmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{bmatrix}}_{\Lambda} \begin{bmatrix} u_1 \\ \vdots \\ u_n \end{bmatrix}$$

2) The solution to the IVP $\begin{cases} \frac{d}{dt}\vec{u} = \Lambda \vec{u} \\ \vec{u}(0) = \vec{u}_0 \end{cases}$

is $\vec{u}(t) = \sum_{k=1}^{n} u_k(0) e^{\lambda_k t} \vec{v_k}$ or $e^{\lambda_k (t-t_0)}$ for $\vec{u}(t_0) = \vec{u}_0$

and $\vec{y}(t) = U \vec{u}(t)$.

Ex 7.12: Solve the IVP

$$\left\{ \frac{d}{dt}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 6x_1 - 20x_2 \\ x_1 - 3x_2 \end{bmatrix}, \quad \begin{bmatrix} x_1(-3) \\ x_2(-3) \end{bmatrix} = \begin{bmatrix} 3 \\ 1 \end{bmatrix} \right)$$

System is $\frac{d}{dt}\vec{x} = A\vec{x}$ where $A = \begin{bmatrix} 6 & -20 \\ 1 & -3 \end{bmatrix}$

Know eigenvectors are $\underset{\lambda_1 = 1}{v_1 = \begin{pmatrix} 4 \\ 1 \end{pmatrix}}$ $\underset{\lambda_2 = 2}{v_2 = \begin{pmatrix} 5 \\ 1 \end{pmatrix}}$ so solution is

$$u(t) = C_1 e^{(t-t_0)} v_1 + C_2 e^{2(t-t_0)} v_2$$

Need $u(-3) = U^{-1}\begin{bmatrix} 3 \\ 1 \end{bmatrix} = \begin{bmatrix} 4 & 5 \\ 1 & 1 \end{bmatrix}^{-1} \begin{bmatrix} 3 \\ 1 \end{bmatrix} =$

$= \begin{bmatrix} -1 & 5 \\ 1 & -4 \end{bmatrix}\begin{bmatrix} 3 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$

so $u(t) = 2e^{(t-3)} v_1 - e^{2(t-3)} v_2$

$$x(t) = \begin{bmatrix} 4 & 5 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 2e^{t-3} \\ -e^{2(t-3)} \end{bmatrix} = \begin{bmatrix} 8e^{t-3} - 5e^{2(t-3)} \\ 2e^{t-3} - e^{2(t-3)} \end{bmatrix}$$

7. iii) More Systems and Examples

Let $\left(\frac{d}{dt}\right)^n x + a_1\left(\frac{d}{dt}\right)^{n-1} x + \dots + a_n x = 0$

be a system of order n.

Ex 7.13 (Companion system, Hamilton's trick)

Set $x_1 = x(t)$
$x_2 = x'(t)$
$\vdots$
$x_n = \left(\frac{d}{dt}\right)^n x(t)$

then $$\frac{d}{dt}\begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} 0 & 1 & & & \\ 0 & 0 & 1 & & \\ & & & \ddots & \\ & & & 0 & 1 \\ a_n & \dots & & a_1 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$$

$\frac{d}{dt}x_1 = x_2$

$\Rightarrow$ an n^{th} order ~~system~~ equation is equivalent to an $n \times n$ firstorder system. (Notice characteristic polynomials in the two different definitions agree).

Ex 7.14: The above trick on $F = ma$ to obtain a first order system

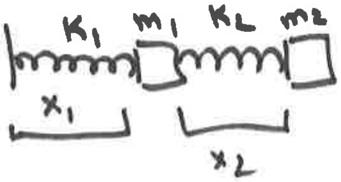
$$\frac{dx}{dt} = F_1(x, p)$$

$p = m\frac{dx}{dt}$
momentum

$$\frac{dp}{dt} = F_2(x, p)$$

gives Hamilton's Equations, the start of Hamiltonian mechanics.

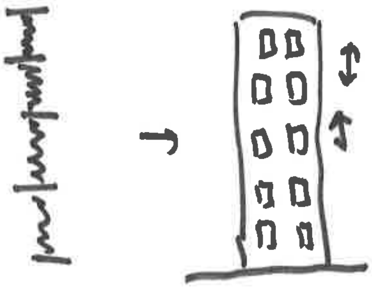
Ex 7.15:



double spring model

$$\left(\frac{d}{dt}\right)^2\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = A\begin{bmatrix} x_1' \\ x_2' \end{bmatrix} + B\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

- used to model car shocks



used to model earthquake resistance of structures.

Lecture 8 First-order Linear Systems II

Recall last time we solve first-order systems

$$\frac{d}{dt} y = A y$$

by decoupling, ie by diagonalizing A. We found the solution when A had real eigenvalues.

Ex 8.1: The matrix $A = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ has no real eigenvalues

- Geometrically it's a rotation, so no vector is stretched
- Algebraically, $\det(A - \lambda Id) = \lambda^2 + 1$ has no real roots.

8.i Diagonalizing with Complex Eigenvalues

Prop 8.2: If we allow $\lambda \in \mathbb{C}$, v_n and A to have entries in $\mathbb{C}$ then diagonalization works the same.

Observe now that the characteristic polynomial now always has roots (by Fundamental Thm of algebra).

Ex 8.3: $A = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ $\quad 0 = \det(A - \lambda Id) = \lambda^2 + 1$

$$\Rightarrow \lambda = \pm i.$$

$$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = i\begin{pmatrix} x \\ y \end{pmatrix} \Rightarrow \begin{matrix} -y = ix \\ x = iy \end{matrix} \Rightarrow v_1 = \begin{pmatrix} i \\ 1 \end{pmatrix}$$

. . .

$$v_2 = \begin{pmatrix} -i \\ 1 \end{pmatrix}.$$

Thm 8.4: The decoupling method still works.

$$\left\{ \frac{d}{dt} y = A y \right\} \xrightarrow{\cdot\,\mathbb{C}} \left\{ \begin{matrix} \frac{d}{dt} y = A y \\ y = C_1 e^{\lambda_1 t} + C_2 e^{\lambda_2 t} \\ \lambda \in \mathbb{C} \end{matrix} \right\} \rightsquigarrow \left\{ y = \underset{\lambda_1}{\mathrm{Re}}\left(C_1 e^{\lambda_1 t} + C_2 e^{\lambda_2 t}\right) \right\}$$

Real system

Complexified system, find eigenvectors

Take real part at end.

Ex 8.5 : Solve

$$\begin{cases} \frac{d}{dt}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 5 & -5 \\ 2 & -1 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \\ x(0) = \begin{pmatrix} 2 \\ 1 \end{pmatrix} \end{cases}$$

Diagonalizing $0 = \det\left(\begin{pmatrix} 5 & -5 \\ 2 & -1 \end{pmatrix} - \lambda \mathrm{Id}\right)$

$$= \lambda^2 - 4\lambda + 5$$

$$\Rightarrow \lambda = \frac{4 \pm \sqrt{16-20}}{2} = 2 \pm i$$

$$\begin{pmatrix} 5 & -5 \\ 2 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = (2+2i)\begin{pmatrix} x \\ y \end{pmatrix}$$

$$\begin{aligned} 5x - 5y &= (2+i)x \\ 2x + y &= (2-i)y \end{aligned} \Rightarrow \begin{aligned} v_1 &= \begin{pmatrix} 3+i \\ 2 \end{pmatrix} \\ v_2 &= \begin{pmatrix} 3-i \\ 2 \end{pmatrix} \end{aligned}$$

$$x(t) = \mathrm{Re}\left[C_1 \underbrace{e^{(2+i)t}}_{e^{2t}(\cos t + i\sin t)} \cdot \begin{pmatrix} 3+i \\ 2 \end{pmatrix} + C_2 e^{(2-i)t}\begin{pmatrix} 3-i \\ 2 \end{pmatrix}\right)$$

$$= \mathrm{Re}\left[C_1 e^{2t}\begin{pmatrix} 3\cos - \sin \\ 2\cos \end{pmatrix} + C_1 i e^{2t} \sim\sim\sim \right.$$

$$C_2 e^{2t}\begin{pmatrix} \cos + 3\sin \\ 2\sin \end{pmatrix} + C_2 i e^{2t} \sim$$

$$= C_1 e^{2t}\begin{pmatrix} 3\cos - \sin \\ 2\cos \end{pmatrix} + C_2 e^{2t}\begin{pmatrix} \cos + 3\sin \\ 2\sin \end{pmatrix}$$

$$t=0 \quad \begin{bmatrix} 2 \\ 1 \end{bmatrix} = x(0) = C_1\begin{bmatrix} 3 \\ 2 \end{bmatrix} + C_2\begin{bmatrix} 1 \\ 0 \end{bmatrix} \Rightarrow \begin{aligned} C_1 &= 1/2 \\ C_2 &= 1/2 \end{aligned}$$

$$x(t) = \begin{bmatrix} e^{2t}(2\cos t + \sin t) \\ e^{2t}(\cos t + \sin t) \end{bmatrix}.$$

Ex 8.6 : See Ex 8.2.7 in book.
Ex 8.2.12

Warning 8.7 : Not all matrices have a basis of eigen~~values~~vectors !

$A = \begin{pmatrix} 1 & c \\ 0 & 1 \end{pmatrix}$ has eigenvalues 1, 1 but

$\begin{pmatrix} 1 & c \\ 0 & 1 \end{pmatrix} - \begin{pmatrix} 1 & \\ & 1 \end{pmatrix} = \begin{pmatrix} 0 & c \\ & 0 \end{pmatrix}$ only has 1-dim nullspace.

8.ii) Solving via the matrix exponential

Recall

$$\frac{d}{dt}y = ky \Rightarrow y = De^{kt}$$

$$\frac{d}{dt}\begin{bmatrix} y \end{bmatrix} = A\begin{bmatrix} y \end{bmatrix} \Rightarrow y = De^{At}$$ formal symbol manipulation.

Def 8.8: The exponential of a matrix A is

$$e^{At} := \mathrm{Id} + At + \frac{A^2t^2}{2} + \frac{A^3t^3}{3!} + \dots$$

Formally, differentiating termwise $y = e^{At}$ solves

$$\frac{d}{dt}y = \frac{d}{dt}\left(\mathrm{Id} + At + \frac{A^2t^2}{2!} + \dots\right)$$

$$= A + A^2t + \frac{A^3t^2}{2} + \dots = A\left(\mathrm{Id} + At + \frac{A^2t^2}{2} + \dots\right)$$

$$= Ay \qquad \checkmark$$

Ex 8.9: If $A = \begin{pmatrix} \lambda_1 & \\ & \lambda_2 \end{pmatrix}$

then

$$e^{At} = \mathrm{Id} + \begin{pmatrix} \lambda_1 & \\ & \lambda_2 \end{pmatrix}t + \frac{\begin{pmatrix} \lambda_1 & \\ & \lambda_2 \end{pmatrix}^2 t^2}{2!}$$

$$= \begin{pmatrix} 1 + \lambda_1 t + (\lambda_1 t)^2/2! + \dots & \\ & 1 + \lambda_2 t + \frac{(\lambda_2 t)^2}{2!} + \dots \end{pmatrix} = \begin{pmatrix} e^{\lambda_1 t} & \\ & e^{\lambda_2 t} \end{pmatrix}.$$

Ex 8.10: If $A = U\Lambda U^{-1}$ $\quad \Lambda = \begin{pmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{pmatrix}$

then

$$e^{tA} = \mathrm{Id} + U\Lambda U^{-1}t + \frac{(U\Lambda \cancel{U^{-1}})(\cancel{U}\Lambda U^{-1})t^2}{2!}$$

$$= \mathrm{Id} + U\Lambda U^{-1}t + \frac{U\Lambda^2 U^{-1}t^2}{2!} + \dots$$

$$= Ue^{t\Lambda}U^{-1}.$$

Thm 8.11: For any nxn matrix, the unique solution of IVP

$$\begin{cases} \frac{d}{dt}[y] = A[y] \\ y(t_0) = [y_0] \end{cases}$$

is

$$[y(t)] = e^{A(t-t_0)}[y_0]$$

Ex 8.12:

$$e^{At}[y_0] = \underbrace{U e^{\Lambda t} \underbrace{U^{-1}[y_0]}_{\text{convert to eigenbasis}}}_{\text{convert back}}$$

reproduces the solution in the diagonalizable case.

Def 8.13: A matrix A w/o a basis of eigenvectors is called <u>non-diagonalizable</u> or "defective".

Important Rem 8.14: Using matrix exponentials and computing e^{At} explicitly is a terrible way to solve.

$A \cdot A = \mathcal{O}(n^3)$ operation

$\Lambda \cdot \Lambda = \mathcal{O}(n)$ operation $\Rightarrow$ diagonalize first.

It is just for conceptual completeness to show there is a solution even if A non-diagonalizable.

Import Rem 8.15: Observe $\boxed{e^{A+B} \neq e^{A} e^{B}}$ in general (true if $AB = BA$)

In particular sol. of

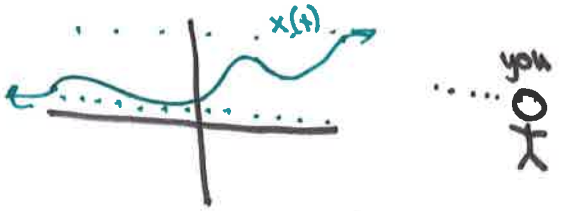
$$\frac{d}{dt}[y] = (A+B)[y]$$

is <u>not</u> $e^{At}e^{Bt}[y]$.

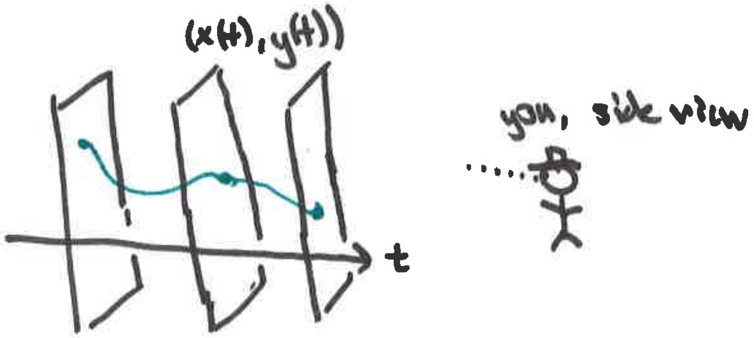
This fact makes life challenging but is also the basis of many interesting things, including Heisenberg's uncertainty principle.

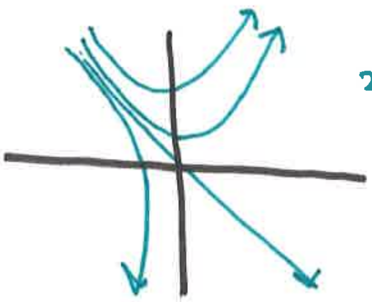
Lecture 9 Phase portraits of linear 2d systems + dynamics

Recall for ODEs (not systems) the qualitative behavior was characterized by the "phase plot"



"Phase diagram/portrait"





2d phase diagram

In two dimensions, the phase diagram can be much more complicated! (ellipses, spirals, saddles, etc).

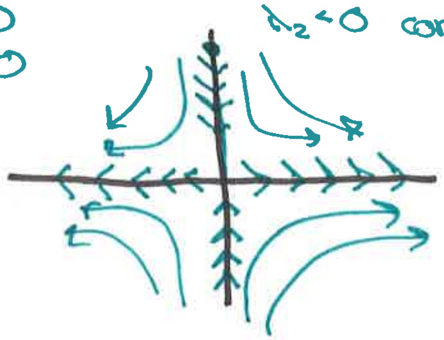
Motivation 9.1: Most interesting equations are non-linear, but understanding linear reductions / approximations is essential.

There are many cases for a 2×2 system, depending on the eigenvalues of A.

9.i) Two real eigenvalues

Ex 9.2: Consider $A = \begin{pmatrix} \lambda_1 & \\ & \lambda_2 \end{pmatrix}$ we know solution is $x(t) = x_0 e^{\lambda_1 t}$, $y(t) = y_0 e^{\lambda_2 t}$

$\lambda_1 > 0$
$\lambda_2 < 0$

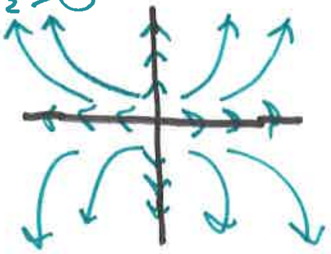


$\lambda_2 < 0$ converges to 0 exponentially

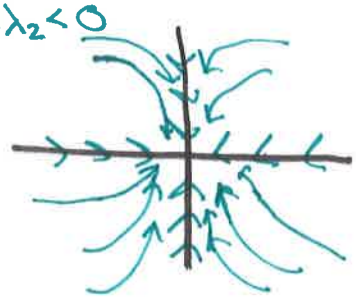
$\lambda_1 > 0$ diverges exponentially

For (x_0, y_0) not on axis, y decreases exp and x increases.

$\lambda_1 > 0$, $\lambda_2 > 0$
$\lambda_2 > \lambda_1$



$\lambda_1 < \lambda_2 < 0$



Def 9.3: If $\lambda_1, \lambda_2 > 0$ the origin is called a <u>source</u> and is unstable (solution curves diverge)

If $\lambda_1, \lambda_2 < 0$ the origin is called a <u>sink</u> and is stable

If $\lambda_1 > 0$ $\lambda_2 < 0$ or reverse the origin is a <u>saddle</u>, and is semi-stable.

The general case is the same, we just squish axes to be the eigenvectors!

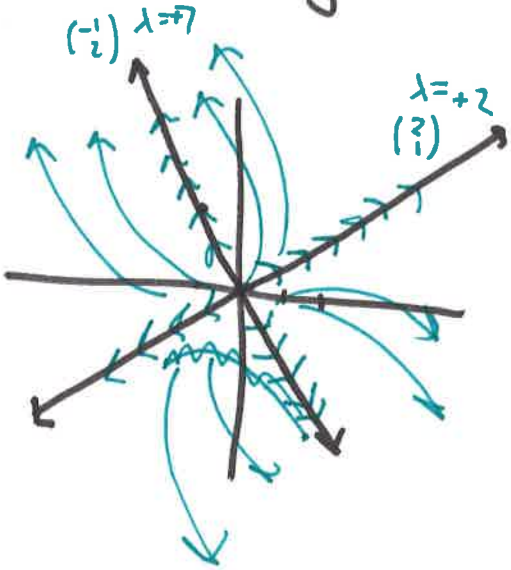
Ex 9.4: Consider $\frac{d}{dt}x = Ax$ for $A = \begin{bmatrix} 3 & -2 \\ -2 & 6 \end{bmatrix}$. Draw phase diagram

$$0 = \lambda^2 - 9\lambda + 14$$

$$\Rightarrow \lambda = 7, 2$$

$$\begin{bmatrix} 3 & -2 \\ -2 & 6 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = 7\begin{bmatrix} x \\ y \end{bmatrix} \Rightarrow v_1 = \begin{pmatrix} -1 \\ 2 \end{pmatrix}$$

$$\vdots\ 2\begin{bmatrix} x \\ y \end{bmatrix} \Rightarrow v_2 = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$$



Ex 9.5: Consider $\frac{d}{dt}x = Ax$ $A = \begin{bmatrix} -4 & 2 \\ 3 & -9 \end{bmatrix}$. Draw the phase diagram

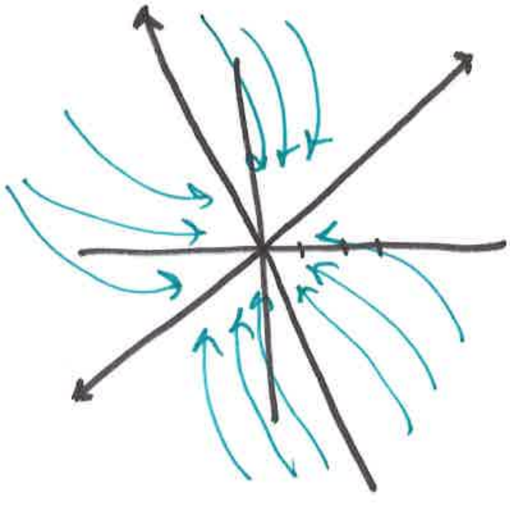
$$0 = \lambda^2 + 13\lambda + 30$$

$$\lambda = -10, -3$$

null

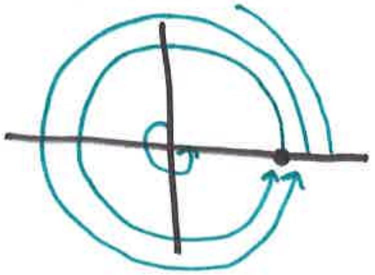
$$\begin{bmatrix} 6 & 2 \\ 3 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = 0 \quad v_1 = \begin{pmatrix} 1 \\ -3 \end{pmatrix} \quad \lambda = -10$$

$$\begin{bmatrix} -1 & 2 \\ 3 & -6 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = 0 \quad v_2 = \begin{pmatrix} 2 \\ 1 \end{pmatrix} \quad \lambda = -3$$



9.1ii) Non-Real Eigenvalues

Ex 9.6 : $A = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ $\quad y(t) = c_1 \begin{pmatrix} \cos t \\ -\sin t \end{pmatrix} + c_2 \begin{pmatrix} \sin t \\ \cos t \end{pmatrix}$



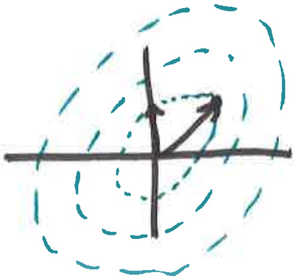
More generally if both eigenvalues are purely imaginary, then the solution curves are ellipses w/ axes the eigenvectors (real parts)

Ex 9.7 : $A = \begin{bmatrix} 1 & -2 \\ 1 & -1 \end{bmatrix}$

$0 = \lambda^2 + 0\lambda + 1 \Rightarrow \lambda = \pm i$

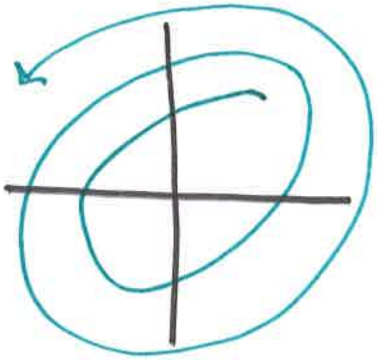
$v_1 = \begin{pmatrix} 1+i \\ 1 \end{pmatrix}$ $\qquad \operatorname{Re} v_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

$v_2 = \begin{pmatrix} 1-i \\ 1 \end{pmatrix}$ $\qquad \operatorname{Im} v_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$



Ex 9.8 : If eigenvalues are $r_0 \pm is$ then the phase diagram is a spiral in or spiral out (depending on the sign of r_0)

$e^{r_0 t}$ (ellipse solution)



Ex 9.9 : Determine Spiral in or Spiral out

$$A_1 = \begin{bmatrix} -1 & 2 \\ -2 & -1 \end{bmatrix} \qquad A_2 = \begin{bmatrix} 1 & 3 \\ -2 & 1 \end{bmatrix}$$

$$0 = \lambda^2 + 2\lambda + 5 \qquad 0 = \lambda^2 - 2\lambda + 7$$

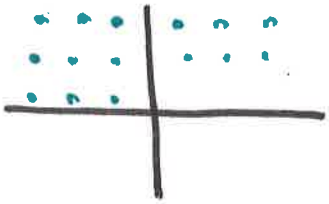
$$\frac{-2 \pm \sqrt{4-20}}{2} = \underbrace{-1}_{} \pm 2i \qquad \lambda = \frac{2 \pm \sqrt{4-28}}{2} = 1 \pm \sqrt{6}i$$

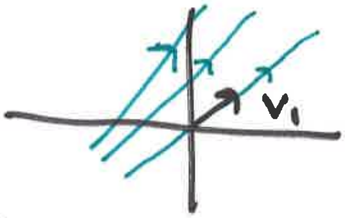
$\Rightarrow e^{-t}$(ellipses) $\qquad \Rightarrow e^{t}$(ellipses)

Spiral in $\qquad$ Spiral out.

9.iii) Exceptional Cases

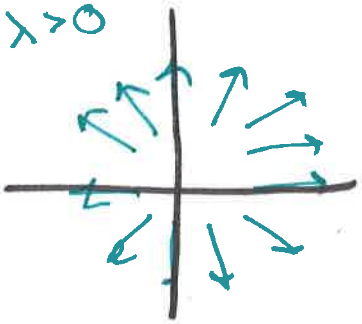
Ex 9.10 : $A = 0$ then solution curves are constant!

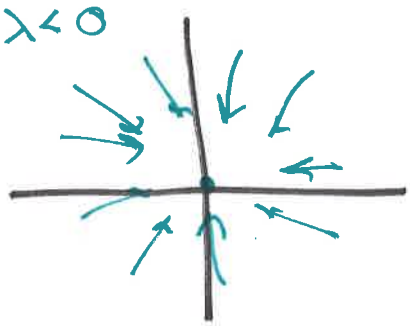




If one zero eigenvalue v_1, then

Ex 9.11 : If repeated real eigenvalues





Lecture 10 Inhomogeneous First-Order Linear ODEs.

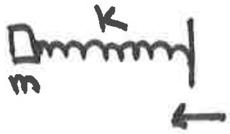
Def 10.1: An ODE

$$a_n(t)\left(\frac{d}{dt}\right)^n y + \dots + a_1(t)\frac{d}{dt}y + a_0(t) = f(t)$$

is said to be homogeneous if $f(t) \equiv 0$, and inhomogeneous if $f \not\equiv 0$.

Rem 10.2: $f(t)$ is called a source term (e.g. heat), a driving term (e.g. springs) or various other thing depending on context.

Ex 10.3: (Resonance) Consider



$f(t) = A\cos(wt)$.

The solution to the driven equation

$$m\frac{d^2x}{dt^2} + Kx(t) = A\cos(wt)$$

exhibits resonance pheomena depending strongly on the frequency of w.

10.i) General and Particular solutions

Recall that for a linear equation $\left[a_n(t)\left(\frac{d}{dt}\right)^n + \dots + a_0(t)\right](y_1+y_2)$ then solutions obey superposition. With inhomogeneous equations

$$a_n(t)y_1^{''\dots'} + \quad a_0(t)y_1 = f_1$$
$$a_n(t)y_2^{(n)} + \dots + a_0(t)y_2 = f_2 \quad +$$

$$a_n(t)\left(\frac{d}{dt}\right)^n(y_1+y_2) + \dots + a_0(t)(y_1+y_2) = f_1+f_2.$$

In particular, if $f_2 = 0$, ie if y_2 is the homogeneous solution then y_1+y_2 solves same inhomogeneous equation w/ f_1.

Thm 10.4: The general solution of the inhomogenous/ driven equation

$$a_n(t)\left(\frac{d}{dt}\right)^n y + \dots + a_0(t)y = f_1(t)$$

is

$$y(t) = y_p(t) + C_1y_1(t) + C_2y_2(t) + \dots + C_ny_n(t)$$
$$= y_p(t) + y_h(t).$$

where • $y_p(t)$ is any particular solution of

$$a_n(t)\left(\frac{d}{dt}\right)^n y_p + \dots + a_0(t) y_p = f_1(t)$$

• $c_1 y_1(t) + \dots + c_n y_n(t)$ are the n linearly independent solutions to the homogeneous equation

$$a_n(t)\left(\frac{d}{dt}\right)^n y + \dots + a_0(t) y = 0$$

For IVP the same logic holds, but now we solve IVP for homogeneous system

Thm 10.5: The solution of the IVP

$$\begin{cases} a_n(t)\left(\frac{d}{dt}\right)^n y + \dots + a_0(t) y = f_1(t) \\ y(t_0) = y_0, \quad y'(t_0) = y_0' \dots \quad y^{(n-1)}(t_0) = y_0^{(n-1)} \end{cases}$$

is $y(t) = y_p(t) + y_h(t)$

where • y_p same as above

• y_h solves the homogeneous IVP to determine c_n,

$$y_p(t_0) + c_1 y_1(t_0) + \dots + c_n y_n(t_0) = y_0.$$

and same for derivatives

Rem 10.6: The same method applies for systems

$$\frac{d}{dt} y = Ay + \vec{f}(t)$$

$$y = [y_p] + [y_h].$$

10.ii) Method of Integrating Factors

We know how to find $y_h(t)$ by solving the homogeneous problem (at least for constant coefficients). There are several methods for finding y_p

Method 10.7 (Integrating Factors)

$$y_p(t) = e^{-at} \underset{\substack{\text{for IVP} \\ \text{(definite int)}}}{\int_{t_0}^{t}} e^{as} f(s)\,ds \quad \text{or} \quad \underset{\text{for general sol}}{\int} e^{at} f(t)\,dt$$

is a particular solution of $y' + ay = f(t)$ for $a \in \mathbb{R}$ constant.

Ex 10.8 : Solve the IVP

$$\begin{cases} y' + 5y = t \\ y(3) = 2. \end{cases}$$

- The (general) solution of the homogeneous problem

$$\begin{cases} y' + 5y = 0 \\ y(3) = 2 \end{cases}$$

is

$$y_h = De^{-5(t-3)}$$

$$y_h(t) = De^{-5(t-3)}.$$

- The particular solution is

$$y_p(t) = e^{-5t} \int te^{5t}\, dt$$

$$= e^{-5t}\left[\frac{1}{5}e^{5t} \cdot t - \frac{1}{5}\int e^{5t}\, dt\right] \qquad u = e^{5t} \quad v = t$$

$$= e^{-5t}\left[\frac{1}{5}e^{5t} \cdot t - \frac{1}{25}e^{5t}\right]$$

so

$$y(t) = \frac{1}{5}t - \frac{1}{25} + De^{-5(t-3)}$$

$$\Rightarrow \quad 2 = y(3) = \frac{3}{5} - \frac{1}{25} + D \qquad D = \frac{36}{25}.$$

Ex 10.9 : Solve $y' + 5y = e^{bt}$

$$y(t) = y_h(t) + y_p(t) = e^{-5t}\int e^{5t}e^{bt}\, dt = \frac{e^{bt}}{b+5} + De^{-5t}.$$

then can solve IVP if we want, but there's an exceptional case.

Notice as $b \to -5$ the solution diverges. This case is exceptional.

$$e^{-5t}\int e^{5t}e^{-5t} = te^{-5t}$$

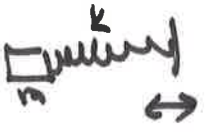
so

$$y(t) = te^{-5t} + De^{-5t}.$$

10.iii) Method of undetermined Coefficients

Recall before we had the "Ansatz" method for constant coefficient higher order equations. In a similar fashion, physical or mathematical intuition can lead to effective ansatzes (ansatzi?) for inhomogeneous system.

Ex 10.10: Consider $\vec{x}''(t) + \frac{k}{m}x(t) = \frac{A}{m}\cos(wt)$ driven spring system. We expect system to eventually align with oscillation. Guess/Ansatz



$$x(t) = C_1\cos(wt) + C_2\sin(wt).$$

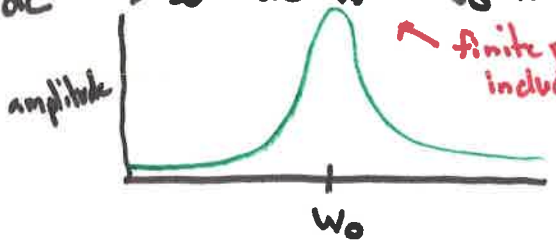
$$x'' + \frac{k}{m}x = -w^2\left(C_1\cos(wt) + C_2\sin(wt)\right) + \frac{k}{m}\left(C_1\cos(wt) + C_2\sin(wt)\right)$$

$$\left(\frac{k}{m} - w^2\right)C_1 = \frac{A}{m} \quad \text{(cos eq)} \qquad \Rightarrow \quad C_1 = \frac{A}{m(w_0^2 - w^2)}$$

$$\left(\frac{k}{m} - w^2\right)C_2 = 0 \quad \text{(sin eq)}$$

$w_0 = \sqrt{\frac{k}{m}}$ is undriven oscillation frequency.

Thus the amplitude $\to \infty$ as $w \to w_0$ is the natural oscillation frequency.



↖ finite peak if we include friction.

Ex 10.11: See Ex 10.3.5 in book.

Ex 10.12: See table 10.3.2. for more useful ansatzes. e.g.

$$f(t) = kt^m \qquad y(t) = A_m t^m + \dots + A_1 t + A_0.$$

Lecture 11 Inhomogeneous Linear ODEs II

Recall we solved the driven spring system with an ansatz. For first order, we can find y_p via integrating factors. Consider now a 2nd order equation

$$\frac{d^2}{dt^2}y + a\frac{d}{dt}y + by = f(t).$$

11.i) More undetermined Coefficients

Ex 11.1: Once we know $y_p(t)$, solving the IVP works same as before:

Solve

$$\begin{cases} \frac{d^2}{dt^2}y - y = t \\ y(0) = 1 \\ y'(0) = 3 \end{cases}$$

given that $y = -t$ is a particular solution.

Solving homogenous, $y = e^{\lambda t}$ $\Rightarrow (\lambda^2 - 1) = 0$ $\lambda = \pm 1$, so

$$y_h = C_1 e^t + C_2 e^{-t}.$$

then

$$y(t) = y_p(t) + y_h(t)$$
$$y(t) = -t + C_1 e^t + C_2 e^{-t}$$
$$y'(t) = -1 + C_1 e^t - C_2 e^{-t}$$

$$\Rightarrow \quad \begin{matrix} 1 = y(0) = C_1 + C_2 \\ 3 = y'(0) = -1 + C_1 - C_2 \end{matrix} \Rightarrow \begin{matrix} C_1 + C_2 = 1 \\ C_1 - C_2 = 4 \end{matrix} \Rightarrow \begin{matrix} C_1 = \frac{5}{2} \\ C_2 = -\frac{3}{2}. \end{matrix}$$

$$y(t) = -t + \frac{5}{2}e^t + (-\tfrac{3}{2})e^{-t}.$$

Ex 11.2: Find the general solution of $y'' + y = t^2$

As before $y_h(t) = C_1 e^{\lambda_1 t} + C_2 e^{\lambda_2 t}$ $\quad 0 = \lambda^2 + 1 \Rightarrow \lambda = \pm i$

$$= C_1 \cos(t) + C_2 \sin(t).$$

Since $f(t)$ is polynomial, guess that $y_p(t)$ is also a polynomial of same order.

$$y_p(t) = At^2 + Bt + C.$$

$$y_p'' + y_p = 2A + (At^2 + Bt + C) = t^2$$

Comparing coeficients

$$A = 1$$
$$B = 0$$
$$2A + C = 0 \quad \Rightarrow \quad C = -2.$$

$$y_p(t) = t^2 - 2$$
$$y(t) = t^2 - 2 + c_1\cos(t) + c_2\sin(t).$$

Rem 11.3: The motivation for the ansatz is that many "classes" of functions (eg polynomial, trig, exponential) are preserved by differentiation, so $f(t) =$ polynomial $\Rightarrow$ $y_p =$ polynomial

Rem 11.4: Within classes of functions, if functions are not multiples then they're linearly independent, this is why e.g.

$$(A)t^2 + (B)t + (2A+C) = 0$$

implies all three coefficients are zero!

Ex 11.5 (Slightly harder) Find the general solution of

$$y'' - 2y' + 3y = 6t^3 - 3t^2 + 3t - 8.$$

homogeneous $y_h = C_1 e^t\cos(\sqrt{2}t) + C_2 e^t \sin(\sqrt{2}t)$ $\quad \lambda^2 - 2\lambda + 3 \Rightarrow \lambda = 1 \pm \sqrt{2}i$

Ansatz $y_p = A_3t^3 + A_2t^2 + A_1t + A_0.$

$$y'' = 6A_3t + 2A_2$$
$$y' = 3A_3t^2 + 4A_2 + A_1$$
$$y = A_3t^3 + A_2t^2 + A_1t + A_0.$$

$$\begin{aligned} y'' - 2y' + 3y &= (6A_3t + 2A_2) - 2(3A_3t^2 + 4A_2 + A_1) + 3(A_3t^3 + \dots + A_0) \\ &= 3A_3t^3 + (-6A_3 + 3A_2)t^2 + (6A_3 - 4A_2 + 3A_1)t + (2A_2 - 2A_1 + 3A_0) \end{aligned}$$

Comparing w/ coefficients of right-hand-side,

$$\begin{aligned} 3A_3 &= 6 \\ -6A_3 + 3A_2 &= -3 \\ 6A_3 - 4A_2 + 3A_1 &= 3 \\ - \quad 2A_2 - 7A_1 + 3A_0 &= -8 \end{aligned} \Rightarrow \begin{bmatrix} 3 & & & \\ -6 & 3 & & \\ 6 & -4 & 3 & \\ & 2 & -2 & 3 \end{bmatrix} \begin{bmatrix} A_3 \\ A_2 \\ A_1 \\ A_0 \end{bmatrix} = \begin{bmatrix} 6 \\ -3 \\ 3 \\ -8 \end{bmatrix}$$

solving $A_3 = 2, \quad A_2 = 3, \quad A_1 = 1, \quad A_0 = -4$

$$y_p(t) = 2t^3 + 3t^2 + t - 4.$$

11.ii) Variation of Parameters : another method of finding y_p.

Consider $y'' + by' + cy = f(t)$.

and suppose $y_h = c_1 y_1(t) + c_2 y_2(t)$

Ansatz 11.6 (Variation of parameters ansatz) Assume $\nearrow$ book uses $u_i = c_i$, $y_i = u_i$ notation

$$y_p(t) = u_1(t) y_1(t) + u_2(t) y_2(t). \qquad \text{and } u_1' y_1 + u_2' y_2 = 0$$

Then

$$y_p'(t) = \underbrace{\cancel{u_1' y_1} \overset{=0}{+} u_1 y_1' + \cancel{u_2' y_2}} + u_2 y_2' = u_1 y_1' + u_2 y_2'$$

$$y_p''(t) = u_1' y_1' + u_1 y'' + u_2' y_2' + u_2 y_2''$$

$$\begin{aligned} y''_p(t) + by_p'(t) + c y_p(t) &= u_1 y_1'' + bu_1 y_1' + cu_1 y_1 + u_2 y_2'' + bu_2 y_2' + cu_2 y_2 \\ &\quad u_1' y_1' + u_2' y_2' + \cancel{bu_1' y_1 + b} = 0 \\ &= u_1 \underbrace{(y_1'' + by_1' + cy_1)}_{=0} + u_2 \underbrace{(y_2'' + by_2' + cy_2)}_{=0} \\ &\quad + u_1' y_1' + u_2' y_2 \end{aligned}$$

by assumption y_1, y_2 solve

$$\Rightarrow u_1' y_1' + u_2' y_2' = f(t)$$

So we find u_1', u_2' satisfy

$$\begin{cases} u_1' y_1 + u_2' y_2 = 0 \\ u_1' y_1' + u_2' y_2' = f(t) \end{cases}$$

Solving + Integrating yields

Thm 11.7 (Variation of Parameters) Assume that the Wronskian

$$W(y_1, y_2) = y_1 y_2' - y_2 y_1' = \det\begin{pmatrix} y_1 & y_2 \\ y_1' & y_2' \end{pmatrix} \neq 0.$$

Then

$$u_1(t) = -\int \frac{y_2(t) f(t)}{W(y_1, y_2)}\, dt \qquad u_2(t) = \int \frac{y_1(t) f(t)}{W(y_1, y_2)}\, dt$$

and the coefficients, and

$$y_p(t) = C_1\left(-\int_{t_0}^{t} \frac{y_2(t) f(t)}{W}\, \underset{ds}{dt}\right) y_1(t) + C_2\left(\int_{t_0}^{t} \frac{y_1(t) f(t)}{W}\, \underset{ds}{dt}\right) y_2(t).$$

Solves IVP $y_p(t_0) = y_p'(t_0) = 0$, integrate def integral, $C_1 = C_2 = 1$

Rem 11.8: This works for $b = b(t)$ $c = c(t)$ also. If constants, then

$$W \neq 0$$

is linear independence, and doesn't have to be checked.

Ex 11.9: $y'' - y = \frac{1}{e^t + 1}$ (not "class" of function unclear so ansatz is hard)

$y_1 = e^t$ $\quad y_2 = e^{-t}$ $\quad W(y_1, y_2) = e^t(-e^{-t}) - e^{-t}e^t = -2.$

$$y_p(t) = C_1 \frac{1}{2}\left[\int e^{-t} \cdot \frac{1}{e^t + 1}\, dt\right] e^t + -\frac{1}{2}\left[\int e^t \cdot \frac{1}{e^t + 1}\, dt\right] e^{-t}$$

$v = e^t + 1$

$$\Rightarrow \int \frac{1}{v(v-1)^2}\, dv \Rightarrow \int \frac{1}{v} - \frac{1}{v-1} + \frac{1}{(v-1)^2}\, dv \Rightarrow \ln|v| - \ln|v-1| - \frac{1}{v-1} + C_1$$

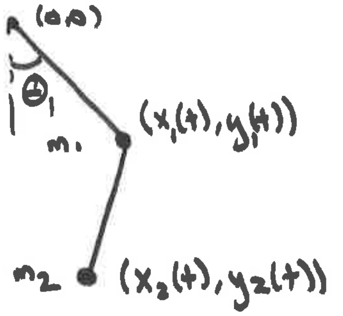
$$\Rightarrow \int e^{-t} \frac{1}{e^t + 1}\, dt = \ln(e^t + 1) - t - e^{-t} + C_1$$

$$\int e^t \frac{1}{e^t + 1} = \ln(e^t + 1) + C_2 \quad \Rightarrow y_p'(t) = \frac{1}{2}\left[(e^t - e^{-t})\ln(e^t + 1) - te^t - 1\right] + C_1 e^t + C_2 e^{-t}$$

Lecture 12 Chaos and Bifurcation.

Consider the initial value problem for the double pendulum

Ex 12.1:



$$\left.\begin{aligned} m_1 x_1'' &= \\ m_1 y_1'' &= \\ m_2 x_2'' &= \\ m_2 y_2'' &= \end{aligned}\right\} \text{many terms involving } \underline{\text{nonlinear}} \text{ combinations of } x_1, y_1, x_2, y_2, x_1', y_1', x_2', y_2'$$

Nonlinear terms arise from trig functions eg $\sin \Theta = \frac{x_1}{L}$

$\cos \Theta = \frac{y_1}{L}$

In terms of Θ_1, Θ_2, terms like $\cos(\Theta_1 - \Theta_2)$ appear.

The trajectory starting at $(x_0, y_0, \tilde{y}_0, \tilde{y}_0)$ diverges from nearby trajectories.

⟹ Video on double pendulum and initial conditions.

Def 12.2: If the solutions of a (non-linear) IVP

$$\begin{cases} \frac{dx}{dt} = f(t, x(t)) \\ x(0) = x_0 \end{cases}$$

on the interval $t \in [0, T_0]$ has the property that

$$\| x_1(T_0) - x_2(T_0) \| \to 0 \quad \text{as } \|x_1(0) - x_2(0)\| \to 0$$

ie if initial conditions are nearby then solutions at time T_0 are nearby, then the system is called <u>stable</u>

Def 12.3 (Informal Def) If the solution at large t as $t \to \infty$ gets arbitrarily far ~~[illegible]~~ from solutions with very close initial conditions, then the system is called <u>chaotic</u>.

(ie. configuration at large t looks random, independent of initial conditions)

Rem 12.4: Chaotic systems are referenced when talking about the "Butterfly effect".

~~Rem 12.5 (Cauchy Stability) For [illegible] the solution~~

~~to the IVP~~

Ex 12.5 (Three body problem) Consider three planets in space

$$m_1 \begin{bmatrix} x_1'' \\ y_1'' \\ z_1'' \end{bmatrix} = \overbrace{F_1(\vec{x}_2, \vec{x}_3, \vec{v}_2, \vec{v}_3)}^{\text{Force from Newtonian gravity}}$$

$$m_2 \begin{bmatrix} \quad \end{bmatrix} = F_2$$

$$m_3 \begin{bmatrix} \quad \end{bmatrix} = F_3$$

If reducing to the complementary 1st order system, we get a coupled, non-linear 18-dimensional system.

⇒ The three body problem is chaotic, and has no closed form solution in general.

Question 12.6: Is the sun-earth-moon system stable? Can a 5 lb meteor impact disrupt the trajectory arbitrarily much? How long would it take?

12.ii) Examples of Stability + Existence

Fortunately, regardless of stability we always know

Thm 12.7 (Existence and uniqueness) The IVP

$$\begin{cases} \dfrac{d\vec{x}}{dt} = f(t, \vec{x}(t)) \\ \vec{x}(t_0) = \vec{x}_0 \end{cases}$$

has a solution on some time interval $(t_0 - \delta, t_0 + \delta)$, and there is only one solution for every $\vec{x}_0$.

Rem 12.8: The only way the interval can fail to extend to $\mathbb{R}$ is if there is a "blow-up" as in previous example of $y(t) = \frac{1}{1-t}$. This cannot happen for linear systems.

Thm 12.8.5 (Cauchy Stability) The unique solution above is stable on any finite interval $[t_1, t_2]$ where it is defined.

$\boxed{12.3}$

Ex 12.9 : Consider $\frac{dx}{dt} = x(t)$ w/ $x(0) = 1$ on the time interval $[-2,5]$.

Let $x_1(0) = K$ be a nearby solution. How close must we take K to 1 so that

$$|x_1(t) - e^t| < .1 \quad \text{at } t = 5 ?$$

$x_1(t) = Ke^t$ so

$$|x_1(t) - e^t| \Big|_{t=5} = e^5|K-1| \Rightarrow |K-1| < e^{-5} .1 = .0067.$$

Rem 12.10 : 1) Observe that while thm 12.8.5 guarantees stability in practice it is often that the stability scale is too small to be practical, (take $t = 50$ in above example)

2) Even though $|x_1(t) - e^t| < 0.1$ at $t = 5$ as $t \to \infty$ it still diverges.

Rem 12.11 : Stability theory is crucial when solving ODEs on computers as these necessarily work in finite precision approximation. If your stability requires $|K-1| <$ float point accuracy of computer you will have a problem!

12.iii) ODEs with parameters + Bifurcation

Def 12.12 : A parameter in an ODE is a constant in the eqation that is determined externally and is information input to the model (by the mathematician) rather than solved for.

Ex 12.13 : K, spring constant

R, pop growth rate

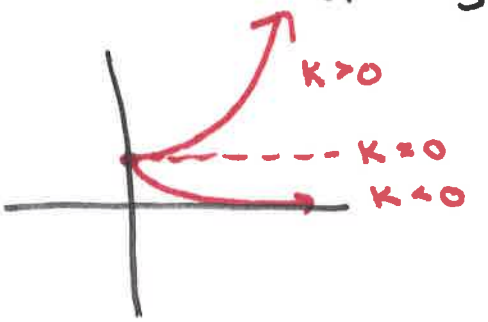
m mass

C, arbitrary const are not! parameters. (solved for not input)

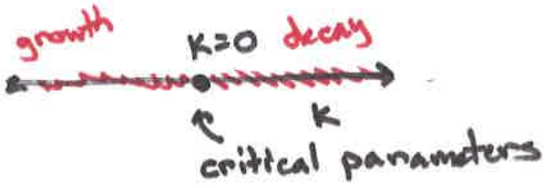
Thm 12.14 : The solutions of ODEs depend continuously (smoothly) on background parameters, over finite intervals.

Def 12.15: A bifurcation occurs when a small smooth change in parameters causes a sudden qualitative change in the behavior of solutions.

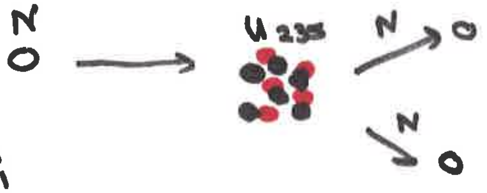
Ex 12.16: Consider $\frac{dy}{dt} = ky$. $x(0) = 1$



At parameter $K=0$ a bifurcation occurs (unbounded growth to staying bounded).



Ex 12.17 (Nuclear Reactors)



In fission reactions, neutrons split uranium nuclei and produce more neutrons

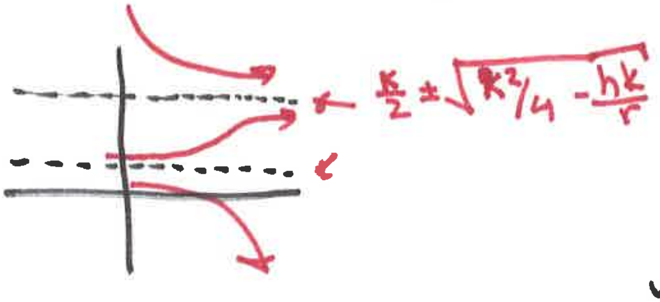
N = free neutrons

$$\frac{dN}{dt} = \alpha N = (\underset{\substack{\uparrow \\ \text{new reaction producing} \\ \text{3 neutrons}}}{\sigma_{fission}} \cdot 3 - \sigma_{absorb} - \sigma_{escape})N.$$

$\alpha > 0$ exp growth (supercritical)
$\alpha = 0$ equil (critical) ✓
$\alpha < 0$ (subcritical).

Ex 12.18: (Logistic Eq w/ harvesting)

$$\frac{dy}{dt} = ry\left(1 - \frac{y}{K}\right) - \underset{\llcorner \text{ constant harvesting rate}}{h}$$



If h is too large so that $f(y_0) < \{$lower stationary sol$\}$
$\Rightarrow$ pop collapse,
when h lowered across that sol, sustainable harvesting can happen.

Lecture 13 Nonlinear Systems I: linearization

Recall a system

$$\vec{x}(t) = F(t, \vec{x})$$

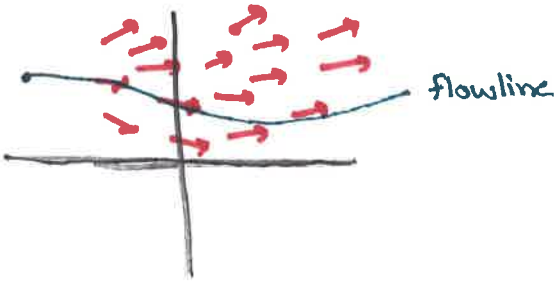
is non-linear if F depends non-linearly on $\vec{x}(t)$.

Ex 13.1: Nonlinear systems of ODEs (sometimes PDEs) occur in

- i) Aero and hydrodynamics
- ii) Newtonian gravity (three body) + General Relativity
- iii) Lokta-Volterra (nonlinear predator-prey)
- iv) models for neuron activity (Hodgkin-Huxley)

Ex 13.2: Recall that a vector field is a function $V: \mathbb{R}^2 \to \mathbb{R}^2$

$$V(x,y) = (f_1(x,y), f_2(x,y))$$



A flowline $\vec{x}(t) = (x(t), y(t))$ is a solution (of $V(x,y)$) of the ODE

$$\nabla \vec{x}(t) = V(x,y)$$

ie

$$\frac{d\vec{x}}{dt} = V(x(t), y(t))$$

(Imagine the vector field as the flow of a river and dropping a leaf at (x_0, y_0) then watching it flow).

Key Observation 13.3: Every non-linear ODE system in n variables is the flow of a vector field on $\mathbb{R}^n$, just set

(possible time dependent)

$$V(x_1 .. x_n) = F(\overset{t,}{x_1}(t), \ldots x_n(t))$$

$\Rightarrow$ Intuition of solution being a flowline applies.

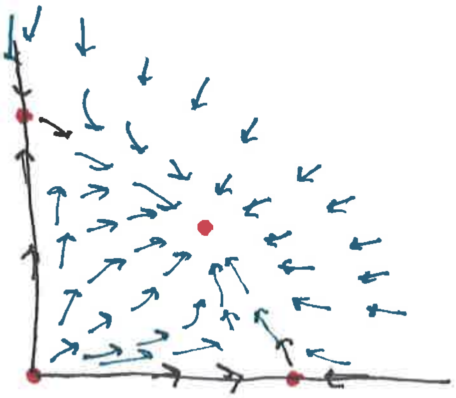
Def 13.4: The phase portrait of the system $\frac{d}{dt}x = F(t,x)$ is a plot of the corresponding vector field, ie the vectors $\vec{F}(t,x)$ at $(x_1 \ldots x_n)$ for many values. A flowline or solution path is $x(t)$ everywhere tangent to these vectors.

Ex 13.5 (~~[illegible]~~ Double Logistic) $P_1(t)$, $P_2(t)$ populations of species

$$\frac{d}{dt}(P_1, P_2) = \left(R_1 P_1(t)\left(1 - \frac{P_1(t) + k_1 P_2(t)}{K_1}\right), R_2 P_2(t)\left(1 - \frac{P_2(t) + k_2 P_2(t)}{K_2}\right)\right)$$

(Logistic model for each w/ k_1, k_2 accounting for species interaction)

These equations have no explicit solutions



Note that $F(t,x) = 0$ at

$(0,0)$

$(0, A_2)$

$(A_2, 0)$

(A_2, A_2)

x_1, x_2 so that

$A_2 := \{ K_2 - x_1 + k_2 x_2 = 0 \}$

$A_1 := \{$ same for x_1 $\}$.

Note that (A_1, A_2) is a sink, $(0, A_2)$ and $(A_1, 0)$ are saddles.

Question 13.6: Can we describe the qualitative behavior near stationary solutions like the above? How do we find these stationary points / equilibria?

Ex 13.7: Many questions about ODEs are actually questions about the local behavior near an equilibrium!

i) Airplanes operate in an equilibrium of aerodynamic Equations, but failures are caused by small unstable deviations from it.

ii) Astrophysic studies stability of certain stationary points of Einsteins equations (black holes).

iii) many chemical / biological system are frequently found near equilibria.

13.ii) Linearized Equations

Recall that we can completely solve constant coefficient linear systems via eigenvectors. Linearization is a process that approximates the non-linear system near an equilibrium by a linear const-coefficient system.

Recall the derivative of $f : \mathbb{R} \to \mathbb{R}$ satisfies

$$f(x+h) = f(x) + h \cdot \frac{df}{dx}(x) + \{\text{error terms } \underset{\text{or } =}{\text{smaller}} \text{ than } h^2\}.$$

For $F : \mathbb{R}^n \to \mathbb{R}^m$

$$F(\vec{x}+\vec{h}) = F(\vec{x}) + J_{\vec{x}}\vec{h} + \{\text{error terms} \sim |h|^2\}$$

where

$$D_xF = J_{\vec{x}} = \begin{pmatrix} \frac{\partial f_1}{\partial x_1} & \cdots & \frac{\partial f_1}{\partial x_n} \\ \vdots & & \vdots \\ \frac{\partial f_n}{\partial x_1} & \cdots & \frac{\partial f_n}{\partial x_n} \end{pmatrix}$$

is the Jacobian Matrix of partial derivatives at $\vec{x}$.

Def 13.8 : A stationary point or equilibrium is a solution

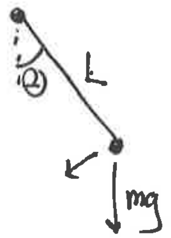
$$F(x_0) = 0$$

At such a point the linearized equations (of the system) is the system (homogeneous const coeff linear)

$$\frac{d\vec{y}}{dt} = \left(D_{x_0} F \right) \vec{y}$$

Rem 13.9 : If $F(x,t)$ is t-dependent can consider the augmented system $x_{n+1} := t$.

Ex 13.10 : Consider the Pendulum : Newton's equations give the nonlinear system

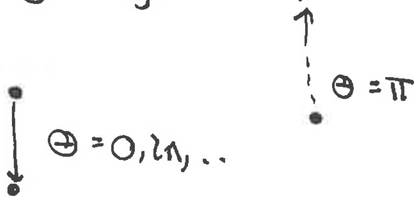


$$\Theta''(t) + \frac{g}{L} \sin \Theta(t) = 0$$

The associated first-order system is $x(t) := \Theta\left(\frac{t}{\sqrt{g/L}}\right)$ $y := x'(t)$

$$\frac{d}{dt}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} y(t) \\ -\sin(x(t)) \end{bmatrix} \quad \text{so} \quad F(x,y) = (y, -\sin x)$$

The linearization at stationary points $(0,0)$
$(\pi, 0)$



The linearization is $$DF = \begin{pmatrix} \frac{\partial}{\partial x} y & \frac{\partial}{\partial y} y \\ \frac{\partial}{\partial x} -\sin x & \frac{\partial}{\partial y} -\sin x \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -\cos x & 0 \end{pmatrix}$$

And the linearized equations are

$$\frac{d\vec{x}}{dt} = \left(D_{(0,0)} F\right) \vec{x} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \vec{x}$$

$$\frac{d\vec{x}}{dt} = \left(D_{(\pi,0)} F\right) \vec{x} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \vec{x},$$

13.iii) Stability Analysis at Critical / Stationary Points

The stability terminology of const coeff linear systems now applies.

Def 13.11 : Suppose DF has real eigenvalues (or non-zero real part)

i) If DF has eigenvalues all with negative real parts at x_0, then x_0 is stable (is a sink)

ii) If at least one eigenvalue has w/ positive real part, then x_0 is unstable (is a source/saddle)

iii) If purely imaginary ~~than~~ eigenvalue exists (and real part ≤ 0 on rest) then x_0 is marginal and there is no general conclusion.

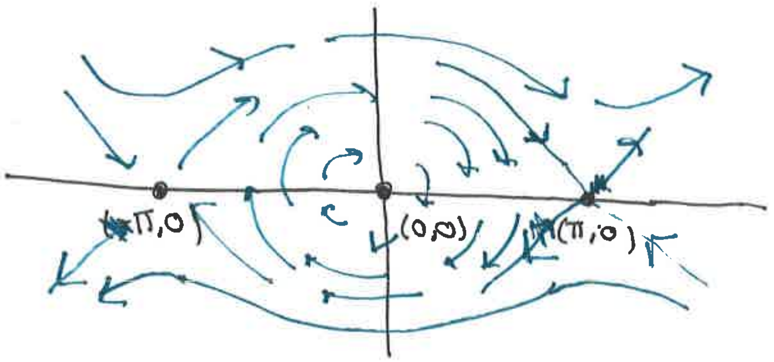
Ex 13.12 : At $(\pi, 0)$ $\det(DF - \lambda Id) = \begin{pmatrix} -\lambda & 1 \\ 1 & -\lambda \end{pmatrix} = \lambda^2 - 1 = (\lambda+1)(\lambda-1)$

$\Rightarrow$ unstable

(note this agrees w/ the physical intuition of instability)

At $(0,0)$ $\det(DF - \lambda Id) = \begin{pmatrix} -\lambda & 1 \\ -1 & -\lambda \end{pmatrix} = \lambda^2 + 1 = (\lambda + i)(\lambda - i)$

$\Rightarrow$ marginal.



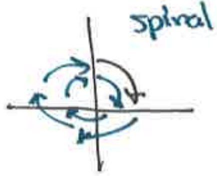
Flow lines in (x,y) phase portrait.

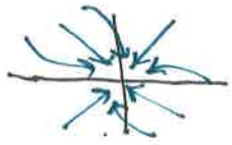
Ex 13.13 : Pendulum w/ friction γ, $DF(x) = \begin{pmatrix} 0 & 1 \\ -\cos x & -\gamma \end{pmatrix}$ $r > 0$.

$\lambda = -\frac{\gamma}{2} \pm \frac{\sqrt{\gamma^2 - 4}}{2}$

Case I $\sqrt{\gamma^2 - 4} \in \mathrm{Im}$ underdamped

Case II $\sqrt{\gamma^2 - 4} > 0$ ~~critically~~ overdamped





Ex 13.14 :

Equilibrium balancing gravity is a Lagrange point. They are unstable (JWST is in such a point).

Lecture 14 Monotone and Conserved Quantities

Consider the ODE system

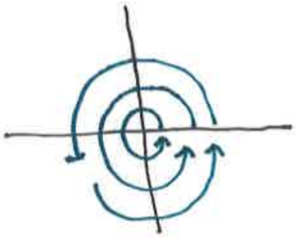
$$\frac{d}{dt}\begin{bmatrix}x\\y\end{bmatrix} = \begin{bmatrix}y + ax(x^2+y^2)\\ -x + ay(x^2+y^2)\end{bmatrix}$$

for $a \in \mathbb{R}$.

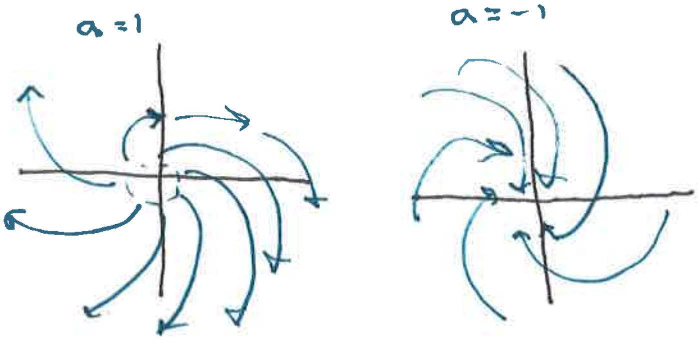
Warning 14.1 : The linearization need to be a good approximation of the nonlinear dynamics in the marginal cases.

Ex 14.2 : $(0,0)$ is clearly a stationary point of the above system. The linearization is

$$D_{(0,0)}F = \begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix} \Rightarrow \lambda = \pm i$$



Dynamics of linearization



Non-linear dynamics

In particular, the linearization cannot distinguish the $a = \pm 1$ cases and their qualitatively different behavior.

Ex 14.3 : Let $d(x,y) = x^2+y^2$ be the distance to the origin. On a solution curve

$d(t) = d(x(t), y(t))$ gives distance as a function of time.

Then

$$\begin{aligned} d'(t) &= 2x(t)x'(t) + 2y(t)y'(t) \\ &= 2x\,(y + ax(x^2+y^2)) + 2y(-x + ay(x^2+y^2)) \\ &= 2a(x^2+y^2)^2 = 2a\,d(t). \begin{matrix}>0\\<0\end{matrix} \quad \text{for } \begin{matrix}a \gtrless 0\\ a<0\end{matrix} \end{aligned}$$

Therefore, distance is always either increasing or decreasing

prevents behavior like this

Def 14.4 : A monotone quantity is a function $V : \mathbb{R}^n \to \mathbb{R}$ such that

$$\frac{d}{dt} V(x(t)) \geq 0 \quad \text{or} \quad \leq 0 \quad \text{along solution curves.}$$

A conserved quantity is one $= 0$.

Ex 14.5: Recall that Newton's equations ⇝ Hamilton's equations (first order system)

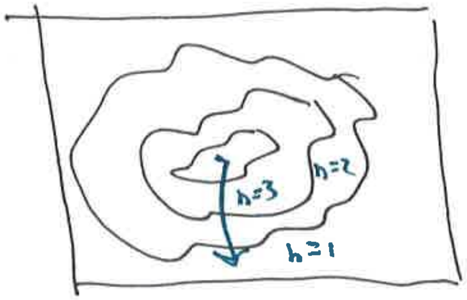
Energy is a conserved quantity (or monotone decreasing w/ friction)
Angular momentum is conserved in systems w/ rotation.

Ex 14.6: (Gradient Descent) Consider $h : \mathbb{R}^n \to \mathbb{R}$, recall the gradient is $\nabla h : \mathbb{R}^n \to \mathbb{R}^n$ $\nabla h = \left(\frac{\partial h}{\partial x_1} \,-\, - \frac{\partial h}{\partial x_n}\right)$ is the direction of fastest increase

$$\frac{dx}{dt} = -\nabla h(x(t))$$

is called the gradient flow (downward) equation.

⇒



By construction,

$$\frac{d}{dt}h(x(t)) = \nabla h(x(t))\cdot x'(t) = -|\nabla h|^2 < 0$$

is monotone

Solution curves cross level sets / contours of h on "topographical ~~map~~" plot

⇒ Monotone quantities : solution curves cross level sets + or −.
Conserved quantities : solution curves stay parallel to level sets.

Rem 14.7: Stationary points are min/max of h. Gradient descent is an important technique in optimization (in particular machine learning).

14.ii) Examples of Monotone + Conserved Quantities

Ex 14.8: Recall the equations for motion of a pendulum (Hamilton's equations)

$$x'(t) = y(t)$$
$$y'(t) = -\sin(x(t))$$

Consider $H(x,y) = \underbrace{mgL\,\frac{y^2}{2}}_{\text{Kinetic}} + \underbrace{mgL(1-\cos(x))}_{\text{potential}} = \text{total energy}$ (H for "Hamiltonian")

14.3

a) show $H(x,y)$ is conserved.

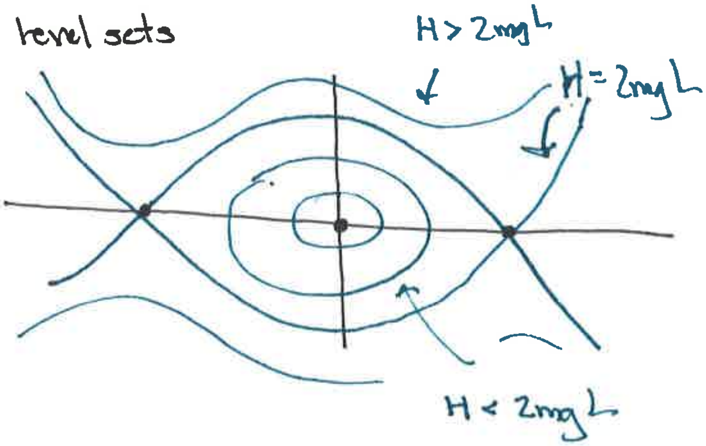
$$\frac{dH}{dt} = \nabla H(x,y) \cdot x'(t)$$

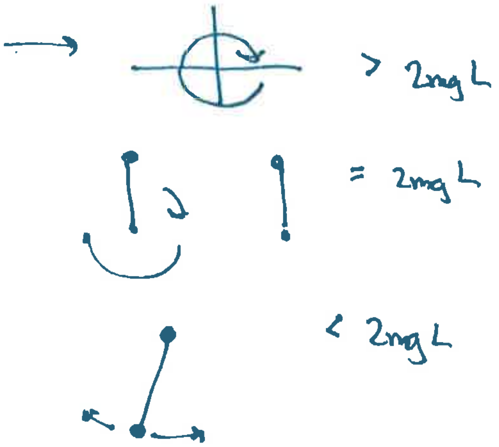
$$= mgL(\sin x, y) \cdot \begin{pmatrix} y \\ -\sin x \end{pmatrix} = mgL(\sin x \cdot y - y \sin x) = 0.$$

b) Find $H(0,0)$, $H(\pi,0)$

$= 0$ $= 2mgL$

c) Plot the level sets





Ex 14.9 : (The SIR model)

Consider a population and the spread of a disease (e.g. flu)

The population can be divided into

$s(t) =$ {fraction of susceptible individuals} S

$i(t) =$ {fraction of infected individuals} I

$r(t) =$ {fraction of removed individuals, immune or dead}. R

The SIR

$$\begin{cases} \frac{ds}{dt} = -I\, s(t)\, i(t) \\ \frac{di}{dt} = I\, s(t)\, i(t) - \gamma\, i(t). \\ \frac{dr}{dt} = \gamma\, i(t) \end{cases}$$

I = infection rate

number of susceptible decreases ~ #infected

∝ infected × suscepti

opposite change as others

γ death + immune recovery rate

Observe that by definition $s + i + r = 1$ is a conserved quantity.

reinfection number $= R_0 = \frac{I}{\gamma}$ average new cases per case.

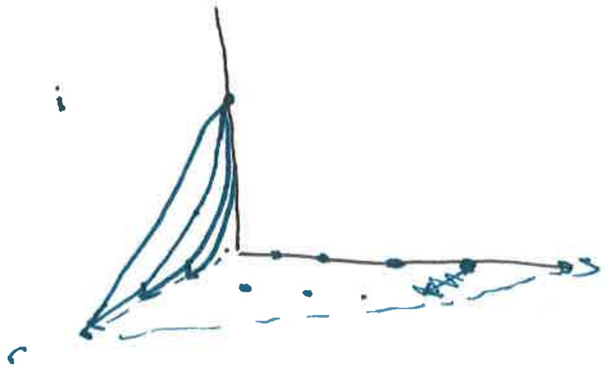
Consider the first two equations

$$\frac{ds}{dt} = -\tau s \cdot i$$
$$\frac{di}{dt} = \tau s \cdot i - \gamma \cdot i$$

i) Every point $(s_0, 0, 1-s_0)$ is a stationary/ constant solution.

Every initial point $(0, i_0, 1-i_0)$ leads to an exponential

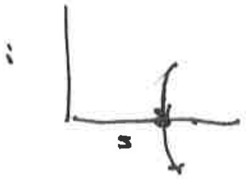
$$\begin{pmatrix} s \\ i \\ r \end{pmatrix} = \begin{pmatrix} 0 \\ i_0 e^{-t\gamma} \\ 1 - i_0 e^{-t\gamma} \end{pmatrix}$$



← stationary on horizontal.

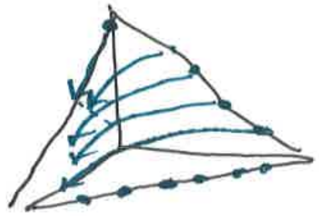
ii) The time evolution keeps all quantities positive

To depart it would have to cross, but by uniqueness, any crossing point would have to be one of the above



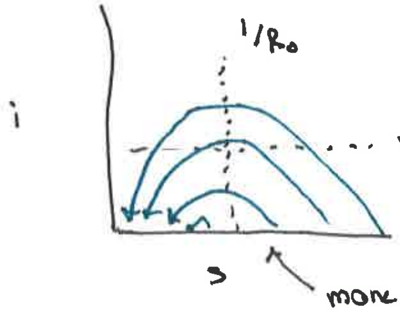
disallowed.

iii) s+i+r stays on the surface of the tetrahedron



iv) Effect of parameters: If $s < \frac{1}{R_0}$ i(t) decreases exponentially

eg. COVID-δ has $R_0 \sim 5$
need 78% vaccinated for exp decay
Covid-o $R_0 \sim 7$ need 85%.



Want to stay below threshhold

more vaccinated at start prevents spread above line

Lecture 15 Power Series Methods

Consider a non-constant coefficient second order equation

$$\frac{d^2y}{dx^2} + a(x)\frac{dy}{dx} + b(x)y = 0$$

Even for $a(x), b(x)$ fairly simple, there are no closed form solutions! Worse for nonlinear. In parts I-II we focused on obtaining qualitative information about the behavior of solutions. Now we focus on the quantitative.

Def 15.1 : A numerical method is an algorithm that can be used to obtain quantitative approximation of solutions of ODEs. The study of such algorithms is called numerical analysis.

Numerical analysis is concerned with questions like

Question 15.2 : Given a numerical method, how close is the approximate solution to the true solution? How does this depend on initial conditions, parameters and the equation?

Question 15.3 : How computationally efficient is the algorithm, i.e. how long will a computer of given processing power require?

Ex 15.4 : (orbital mechanics) Reentry of space shuttle involved numerical solutions to

$$y'' - k_2 e^{-x} y' + (k_1 e^{-x} - k_3 e^{x})y = 0.$$

Accuracy here is important!

Ex 15.5 : (Fluid dynamics) Much of engineering design work for aircrafts, involves solving numerical models of hydrodynamic / Fluid flow equations

Ex 15.6 ITER Divertor. Months of computation for only ~10%-20% accuracy.

$\Rightarrow$ Having a computer does not make getting the answer easy!

Some more concrete examples

Ex 15.7 : $y'' - 2xy' + 2\lambda y = 0$ (Hermite equation, quantum Harmonic oscillator)

Ex 15.8 : $y'' - xy = 0$ (Airy Equation, accoustics / waves)

Ex 15.9 : $r^2 u''(r) + r u'(r) + (\alpha^2 \pm r^2)u(r) = 0$ (Bessel's equation)

The latter appears in modeling the vibrations of a drum (radial direction)

⚠ 15.10: Even the simplest non-constant coefficient equation in 15.8 doesn't have a closed form solution.

The existence + uniqueness theorem still guarantees solutions

Def 15.11: A Special Function is a function defined as the solution to a common ODE (which has no simple expression by common functions)

Ex 15.12: The two linearly independent solutions of

$y'' - xy$ are $Ai(x)$, $Bi(x)$ Airy functions

$r^2 u'' + r u' + (\alpha^2 + r^2) u$ $J_\alpha(r)$, $Y_\alpha(r)$ Bessel functions

$\Rightarrow$ Even computing e.g. $Y_\alpha(1)$ is extremely difficult!

Ex 15.13: Time-dependent diagonalization just doesn't work (see Rem 15.15)

15.i) Review of Power Series

A Power series method is a numerical method based on recursively solving terms in power series expansions of the solution (Recall undetermined coeff)

Recall that a sequence is a collection $\{a_n\} \in \mathbb{R}$ (or $\mathbb{C}$) of numbers, and a series is $\sum_{n=0}^{\infty} a_n$. It is said to converge if the partial sums

$$\lim_{N \to \infty} \sum_{n=0}^{N} a_n = \text{finite}$$

and diverge if it is infinite.

Ex 15.14: If $r < 1$

$$\frac{1}{1-r} = 1 + r + r^2 + \ldots = \sum_{n=0}^{\infty} r^n \quad \text{converges}$$

$\forall x \in \mathbb{R}$

$$\sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2!} + \ldots = e^x \quad \text{converges}$$

Ex 15.15: $\sum_{n=0}^{\infty} \frac{1}{n} = \infty$ diverges $\quad \sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$ converges.

Def 15.16 : A power series is a series depending on x,

$$p(x) = \sum_{n=0}^{\infty} C_n x^n$$

or centered at a $= \sum_{n=0}^{\infty} C_n (x-a)^n$.

The radius of convergence is R if the series converges for $|x-a| < R$
(the largest R)

Thm 15.17 (Taylor) : If $P(x) = \sum_{n=0}^{\infty} C_n (x-a)^n$ then radius of convergence is

$$R = \lim_{n\to\infty} \frac{C_n}{C_{n+1}} < \infty .$$

If $f(x)$ is a function (w/ ∞-many derivatives)

$$P(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!} (x-a)^n$$

is called the Taylor series of f, and $f(x) = P(x)$ for $x \in [a-R, a+R]$

⚠ may have $R=0$ $e^{-1/x}$.

15.iii) Power Series Methods :

This method is just undetermined coefficients w/ ∞-many unknowns (or large N for approximation)

Ansatz 15.18 : The solution $y(t)$ of a given equation has a convergent power series.

$$y(t) = \sum_{n=0}^{\infty} C_n t^n$$

$$y'(t) = \sum_{n=0}^{\infty} n C_n t^{n-1}$$

(differentiating termwise by power rule)

Ex 15.19 : Consider $y'(t) = k y(t)$ $y(0) = 1$

$$y(t) = \sum_{n=0}^{\infty} C_n t^n$$

$$y'(t) = \sum_{n=0}^{\infty} n C_n t^{n-1} \Rightarrow \sum_{n=0}^{\infty} n C_n t^{n-1} = k \sum_{n=0}^{\infty} C_n t^n$$

$$[c_1 + 2c_2 t + 3c_3 t^2 + ..] = k[c_0 + c_1 t + c_2 t^2 + \ldots)$$

By linear independence of t^n, coefficients must be equal for all n!

$$C_1 = KC_0$$
$$2C_2 = KC_1$$
$$3C_3 = KC_2$$

At = 0 $\quad y(t) = C_0$ so $\quad C_1 = Ky_0$

$$2C_2 = K \cdot K y_0 \Rightarrow C_2 = \frac{k^2}{2} y_0$$
$$3C_3 = k \cdot \frac{k^2}{2} \cdot y_0 \Rightarrow C_3 = \frac{k^3}{2 \cdot 3} y_0$$

$\Rightarrow C_n = \dfrac{K^n}{n!} y_0$ $\qquad$ so $\quad y(t) = Ke^{t \cdot k}$.

Ex 15.20 : Solve $\dfrac{dy}{dx} = \dfrac{y}{x} + \dfrac{2(x-1)}{x}$ $\qquad y(1) = a$.

$$((x-1)+1)\frac{dy}{dx} = y + 2(x-1)$$

$y = \sum_{n=0} C_n (x-1)^n$

$y' = \sum_{n=0} nC_n (x-1)^{n-1}$

$$((x-1)+1)y' = (x-1)\sum C_n (x-1)^{n-1} n + \sum C_n (x-1)^{n-1} n$$
$$= \sum_{n=0} C_n (x-1)^n n + \sum_{n=0} C_n (x-1)^{n-1} n$$
$$= C_1(x-1) + 2C_2(x-1)^2 + \ldots$$
$$+ C_1 + C_2 \cdot 2(x-1) + \ldots$$
$$= C_1 + \sum_{n=1}^{\infty} (nC_n + (n+1)C_{n+1})(x-1)^n$$

$$-y - 2(x-1) = -C_0 - (C_1 + 2)(x-1) + \sum_{n=2}^{\infty} C_n (x-1)^n$$

so $0 = (C_1 - C_0) + [(C_1 + 2C_2) - (C_1 + 2)]x - 1 + \sum_{n=2}^{\infty} nC_n + (n+1)C_{n+1} - C_n$

$0 = C_1 - C_0$

$0 = 2C_2 - 2$

$0 = (n-1)C_n + (n+1)C_{n+1}$

$\Rightarrow$ $C_0 = a$

$C_1 = a$

$C_2 = 1$

$C_{n+1} = \dfrac{-(n-1)}{(n+1)} C_n$

$C_3 = -\frac{1}{3}C_2 = -\frac{1}{3}$

$C_4 = \frac{2}{4} \cdot \frac{1}{3} \ldots$

$\boxed{C_n = (-1)^n \frac{2}{n(n-1)}}$ $\uparrow$

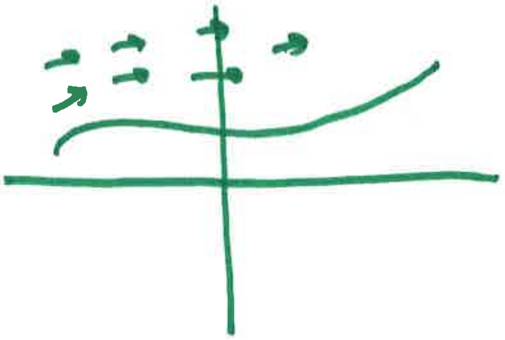
Lecture 16 Numerical Methods II: Euler's Method.

Recall numerical methods are algorithms for approximating solutions of ODEs on computers. We are concerned with their accuracy and efficiency.

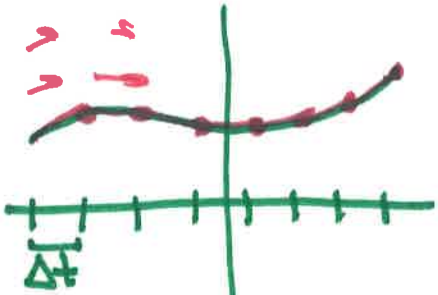
Motivation 16.1: Power series methods are often impractical because they are
i) require solving large systems
ii) require knowledge of equation functions + their Taylor Series.

16.i) Explicit Euler

Consider an ODE $\frac{dx}{dt} = F(x,t)$. The solution of the IVP $x(t_0) = x_0$ is the solution curve in the slope field



$\Rightarrow$



Approximate $x(t)$ by a piecewise linear function on small intervals of width Δt. Set $t_n = t_0 + n\Delta t$.

$$\frac{dx}{dt} \approx \frac{x_{n+1} - x_n}{\Delta t}.$$

$$\Rightarrow x_{n+1} = x_n + \Delta t \frac{dx}{dt}$$
$$= x_n + \Delta t \cdot F(x_n, t_n).$$

Algorithm 16.2 (Explicit Euler)

- Fix Δt small, and initial value x_0.
- For $n = 1, \dots, \frac{T}{\Delta t}$, set
$$x_{n+1} = x_n + \Delta t \cdot F(x_n, t_n).$$

Remark 16.3: Also called "forward Euler".

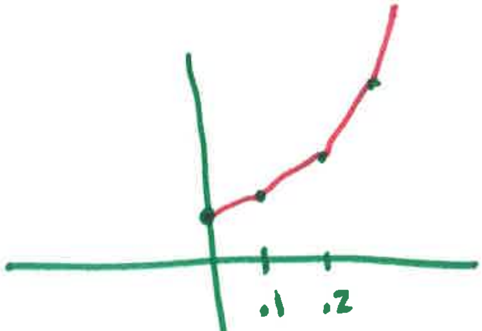
Ex 16.4: $\frac{dx}{dt} = 3x$, $x(0) = 1$. Take $\Delta t = \frac{1}{10}$, $[0,T] = [0,3]$.

$x_0 = 1$
$x_1 = 1 + \frac{1}{10} \cdot 3x_0 = 1 + \frac{3}{10} = 1.3$
$x_2 = 1.3 + \frac{1}{10} \cdot 3(1.3) = 1.3 + .39 = 1.69$

$$x_3 = 1.69 + \tfrac{1}{10} \cdot 3 \cdot 1.69 \approx 2.197$$

$$\vdots$$

$$x_{30} = 2619.99564$$



Check: $x(3) = e^{3 \cdot 3} = 8103.0839$:(

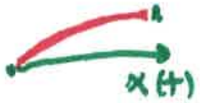
Try $\Delta t = 0.01$	~~x_{end}~~ $x_{300} = 7098.513$	err $=$	-1004.57
$= .001$	$x_{3000} = 7994.6427$	$=$	-108.44
$= 10^{-4}$	$x_{end} = 8092.154$	$=$	-10.9295
$= 10^{-5}$	$x_{end} = 8101.99$	$=$	-1.0938
$= 10^{-6}$	$x_{end} = 8102.974$	$=$	$-.10939$

$\vdots$

16.ii) Accuracy of Explicit Euler

Recall that near x

$$f(x+h) = f(x) + hf'(x) + \mathcal{O}(h^2).$$



error per timestep $\approx (\Delta t)^2$.

Def 16.5: the local truncation error is the error from a single timestep.

Def 16.6: the global error is the maximum of

$$|x_i - x(t_i)|$$

over the interval $[0,T]$.

Thm 16.7: The approx. sol of $x'(t) = F(x,t)$ over $[0,T]$ ~~of~~ obtained by explicit Euler has global error w/ timestep Δt

$$\max |x_i - x(t_i)| \leq \overset{\text{depends on } F, T}{C} \cdot \Delta t.$$

because we take $\frac{T}{\Delta t}$ steps w/ error $\sim (\Delta t)^2 \sim T \cdot \Delta t$.

Generalization 16.8: For systems, all same but vectors,

$$\vec{x}_{n+1} = \vec{x}_n + \Delta t \cdot F(\vec{x}_n, t).$$

16. iii) Higher Order Methods

Idea 16.9 : To get better accuracy, we can use information about higher derivatives

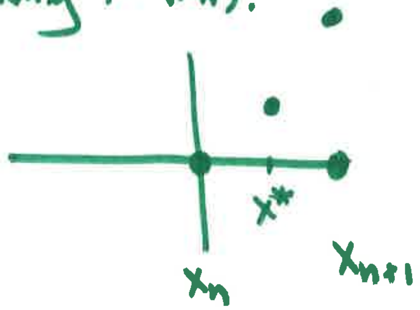
By Taylor's theorem

$$x_{n+1} = x_n + \Delta t F(x_n) + \frac{1}{2}(\Delta t)^2 F'(x_n) + \mathcal{O}(h^3).$$

$$= x_n + (\Delta t)\left[a F(x_n) + b F'(x_n)\right]$$

⇒ Requires knowing $F'(x_n)$.

Instead



⇒ approx. slope at midpoint

$$x_{n+1} = x_n + \Delta t f(x^*)$$

$$= x_n + \Delta t f\left(x_n + \tfrac{1}{2}\Delta t f(x_n)\right).$$

Δt	error
.01	10.6879
10^{-3}	.10914
10^{-4}	.00109 ...

Thm 16.10 : This method has global error

$$\max |x_n - x(t_n)| \leq C \cdot (\Delta t)^2.$$

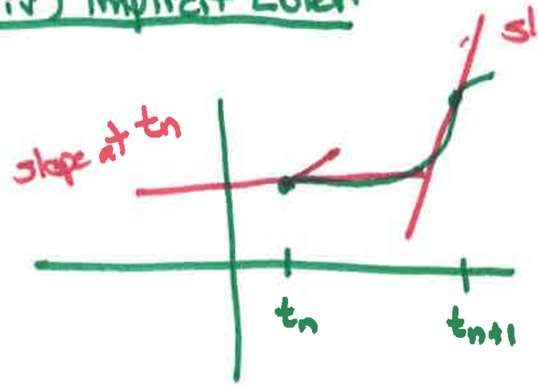
Def 16.11 : If a method has global error $\sim (\Delta t)^k$ it is called a first/second/k^{th} order method.

Important terminology 16.12 : this method is explicit because each stage is calculated from the previous stage only by plugging in!

$$x_{n+1} = x_n + \Delta t f(x_n)$$

$$= \underbrace{x_n + \Delta t f\left(x_n + \tfrac{1}{2} f(x_n)\right)}_{\text{only } x_n \text{ appears on RHS.}}$$

16.iv) Implicit Euler



Could do same but advance by the slope at the point we hit.

$$\frac{x_{n+1} - x_n}{\Delta t} = F(x_{n+1}, t_{n+1}).$$

$$x_{n+1} = x_n + \underbrace{\Delta t \cdot F(x_{n+1}, t_{n+1})}_{\text{cannot just plug in.}}.$$

$$\left[x_{n+1} - \Delta t\, F(x_{n+1}, t_n + \Delta t)\right] = x_n$$

⇒ Need to solve nonlinear equation for each step.

Def 16.13: A method where each step requires solving (not just plugging in) is call *implicit*.

Algorithm 16.14 (Implicit Euler)

- Fix Δt small, initial x_0
- For $n = 1, \dots \frac{T}{\Delta t}$ solve

$$\left[x_{n+1} + \Delta t\, F(x_{n+1})\right] - x_n = 0.$$

Set x_{n+1} = solution.

Lecture 17 Numerical Methods III : Stiff ODEs + Stability

(on python notebook)

Ex 17.1 : Instability of $e^{-\lambda t}$

Ex 17.2 : Instability of systems

Ex 17.3 : Leveque example

$$\frac{dy}{dt} = -\lambda(y - \cos t) - \sin t$$

17.i) The stability threshold

Consider

$$\frac{dy}{dt} = -\lambda y. \qquad \lambda > 0.$$

Then solving w/ explicit Euler shows

$$y_{n+1} = y_n + -\lambda y_n \cdot \Delta t$$

$$= (1 - \lambda \Delta t) y_n.$$

$$\Rightarrow \qquad y^N = (1 - \lambda \Delta t)^N y_0$$

← grows exponentially unless $|1 - \lambda \Delta t| < 1$.

so where

$$1 - \lambda \Delta t < -1$$

$$-\lambda \Delta t < -2$$

$$\Delta t > \frac{2}{\lambda} \qquad \text{exponential growth.}$$

Def 17.4 : the requirement that

$$\boxed{\Delta t \leq \frac{2}{\lambda}}$$

is called the stability threshold (of explicit Euler).

For a diagonal system

$$\frac{dy}{dt} = \begin{pmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{pmatrix} y$$

the stability threshold is $\Delta t \leq \frac{2}{|\max \lambda_i|}$.

In contrast, implicit Euler has

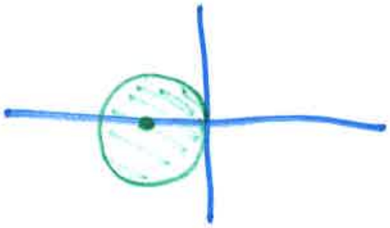
$$y_{n+1} = y_n - \lambda \Delta t \cdot y_{n+1}$$

$$(1 + \lambda \Delta t) y_{n+1} = y_n \implies y_N = (1 + \lambda \Delta t)^{-N} y_0$$

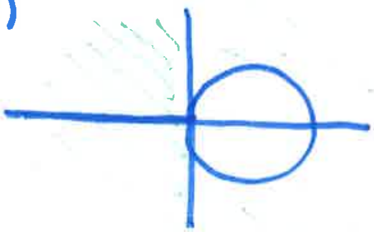
decays exponentially (as it should) for any Δt.

Def 17.5 : The (absolute) stability region of a numerical method is the region of the complex plane such that if $\lambda_i \Delta t$ lies in that region, then erroneous exp. growth does not occur.

Ex 17.6 : (Explicit Euler) $|1 - \lambda \Delta t| < 1 \implies$



Ex 17.7 (Implicit Euler)



negative eigenvalues are important.

Crucial Fact of life 17.8 : Most equations from physics/engineering/real life have negative eigenvalues.

(this is kind of the 2nd law of thermodynamics, see Heat Eq in two weeks).

Comparison 17.9

Implicit	Explicit
absolute stability (Δt any)	potentially unstable, $\Delta t \leq \frac{2}{\lambda}$
Single large timestep more expensive.	Easy, fast individual timesteps

Expensive large timesteps usually win, eg

$\Delta t = 10^6 \times$ bigger

$10^3 \times$ more expensive.

17.ii) The Stiff ODEs

If $$\frac{dy}{dt} = \left(\quad F \quad \right) y \quad \text{and} \quad DF = U \begin{pmatrix} \lambda_1(t) & & \\ & \ddots & \\ & & \lambda_n(t) \end{pmatrix} U^{-1}$$

then need $\Delta t \le \frac{2}{\max \lambda_n(t)}$ for stability, even if we care mostly about effects in small eigendirections.

Def 17.10: An ODE is said to have stiffness ratio

$$S_{ODE} = \left| \frac{Re(\lambda_{max})}{Re(\lambda_1)} \right|.$$

(system of springs where some have high spring constants, ie are stiff).

and an ODE is said to be stiff if S_F is large.

Physical Interpretation 17.11

Stiffness means that that the system has an overlay of physical effects on disparate timescales,

need $\Delta t \le \frac{2}{\lambda_{max}}$

smaller than the fastest!.

Ex 17.12: Van de Pols eq.

$$y'' - \mu(1-y^2)\frac{dy}{dt} + y$$

has
$$x' = y$$
$$y' = \mu(1-x^2)y - x$$

has stiffness $O(\mu)$.

Ex 17.13: The ITER Divertor.

Lecture 18 PDEs I: Definitions and Basics

An Ordinary differential equation relates the ordinary derivatives

$$\left(\frac{d}{dx}\right)^n f + F\left(x, \frac{\partial f}{\partial x}, \frac{\partial^2 f}{\partial x^2}, \dots\right) = 0 .$$

Single Variable $\frac{d}{dx}$ ⟶ Ordinary Differential Equation

↓

Multivariable $\frac{\partial}{\partial x}, \frac{\partial}{\partial y}$ ⟶ <u>Partial Differential Equation</u> (PDE).

<u>Key Idea 18.1</u>:
- PDEs are much harder and much more useful than ODEs (even w/ computers)
- Humans do not really know how
  they only know how to invent increasingly clever methods of
  reducing the problem to solving many ODEs.

18.i) Examples of PDEs

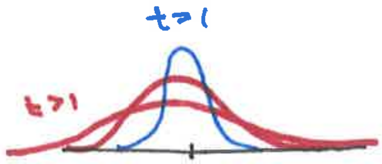
<u>Ex 18.2</u>: (Heat Equation) This is a refinement of Newton's law of cooling

$U(t,x)$ temperature as a function of t time in $\mathbb{R}$
x position in $\mathbb{R}$ (or $[0,L]$).

Then

$$\frac{\partial U}{\partial t} = \alpha \frac{\partial^2 U}{\partial x^2} \qquad \text{(heat eq)}$$

where α is the thermal diffusivity



At $t = 1$ if $U(1,x) = \frac{1}{2\sqrt{\pi}} e^{-x^2/4}$

Then

$$U(t,x) = \frac{1}{(4\pi)^{1/2}} \frac{1}{\sqrt{t}} e^{-x^2/4t}$$

solves the heat eq for $t \geq 1$. $\alpha = 1$

$$\frac{\partial U}{\partial t} = \frac{1}{(4\pi)^{1/2}}\left[-\frac{1}{2}\frac{1}{t^{3/2}} e^{-x^2/4t} + \frac{1}{\sqrt{t}} \cdot -\frac{x^2}{4} \cdot -\frac{1}{t^2} e^{-x^2/4t}\right] = \frac{1}{(4\pi)^{1/2}}\left[\frac{x^2}{4}\frac{1}{t^{5/2}} - \frac{1}{2t^{3/2}}\right] e^{-x^2/4t}$$

$$\frac{\partial U}{\partial x} = \frac{1}{(4\pi)^{1/2}}\left[\frac{1}{\sqrt{t}} \cdot -2x \cdot \frac{1}{4t} e^{-x^2/4t}\right]$$

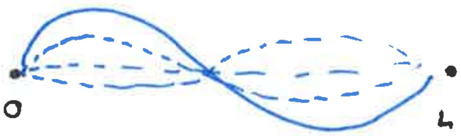
$$\frac{\partial^2 U}{\partial x^2} = \frac{1}{(4\pi)^{1/2}}\left[-\frac{1}{2t^{3/2}} + \frac{(2x)^2}{(4t)^2}\frac{1}{\sqrt{t}}\right] e^{-x^2/4t}$$

$\longleftrightarrow$ $=$ ✓.

<u>Rem 18.3</u>: In 3d

$$\frac{\partial U}{\partial t} = \alpha\left(\frac{\partial^2 U}{\partial x^2} + \frac{\partial^2 U}{\partial y^2} + \frac{\partial^2 U}{\partial z^2}\right), \quad U = \frac{1}{t^{3/2}(4\pi)^{3/2}} e^{-(x^2+y^2+z^2)/4t}$$

Ex 18.3 (Wave Equation) Consider a fixed string



The wave equation

$$\frac{\partial^2 U}{\partial t^2} = c^2 \frac{\partial^2 U}{\partial x^2}.$$

c: propagation speed
constant.

$$U = \sin\left(\frac{2\pi c}{L} t\right) \sin\left(\frac{2\pi}{L} x\right)$$

$$\frac{\partial^2 U}{\partial x^2} = \sin(-)\sin\left(\frac{2\pi x}{L}\right) \cdot -\left(\frac{2\pi}{L}\right)^2 \cdot c^2$$

$$\frac{\partial^2 U}{\partial t^2} = -\sin\left(\frac{2\pi c}{L} t\right)\sin\left(\frac{2\pi x}{L}\right) \cdot -\left(\frac{2\pi c}{L}\right)^2$$

=

In 3d (or 2d)

$$\frac{\partial^2 U}{\partial t^2} = c^2\left(\frac{\partial^2 U}{\partial x^2} + \frac{\partial^2 U}{\partial y^2} + \frac{\partial^2 U}{\partial z^2}\right)$$

Rem 18.4:
- Electromagnetic waves satisfy the wave equation (Maxwell) where $E = U$ = electric field, c = speed of light
- Hydro/aerodynamic equations become wave equation in some situations (ripples on a pond, sound)
- Einstein's Equations in General Relativity are a system of 10 coupled non-linear wave equation. $U = g_{ij}$ measures stretching/bending of spacetime in (t, x, y, z).

Ex 18.5 (Schrödinger Equation)

$$i\hbar \frac{\partial U}{\partial t} = \frac{-h^3}{m}\left[\frac{\partial^2 U}{\partial x^2} + \frac{\partial^2 U}{\partial y^2} + \frac{\partial^2 U}{\partial z^2}\right] + V(x,y,z)\cdot U(t,x,y,z)$$

← potential energy

U is $\mathbb{C}$-valued here.

Ex 18.6 (Transport Equation) U = density of solvent in fluid

$$\frac{\partial U}{\partial t} + v(x,t)\frac{\partial U}{\partial x} = 0$$

↑ flow velocity.

Ex 18.7 (Black-Scholes Equation) V value of stock option at time t price S

$$\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + rS\frac{\partial V}{\partial S} - rV = 0. \qquad r = \text{risk}.$$

18.ii) Linearity + Superposition

Def 18.8: A PDE (of order n) is a function P and the associated equation.

$$P\left(t, x, U, \frac{\partial U}{\partial x}, \frac{\partial U}{\partial t}, \dots \frac{\partial^n U}{\partial x^n}, \frac{\partial^{n-1}\partial U}{\partial x^{n-1}\partial t} \dots \frac{\partial^n U}{\partial t^n}\right) = 0 \quad \doteq f(t,x) \text{[illegible]}$$

Homogenous or inhomogeneous apply as before.

Def 18.9: The PDE is linear if it is a linear function of all derivatives.

Ex 18.10: Heat + Wave equations, Schrödinger, Transport, Black-Scholes are all linear;

If P is linear

$$P = \sum_{j+k=n} \alpha_{jk}(t,x)\frac{\partial^j \partial^k U}{\partial t^j \partial x^k} + \sum_{j'+k'=n-1} \beta_{jk}(t,x)\frac{\partial^{j'+k'} U}{\partial t^{j'} \partial x^{k'}} + \dots + \Omega(t,x)U = 0.$$

Ex 18.11: (minimal surface equation) Soap bubble minimizes area for given boundary

U = graph over (x,y) in plane.

$$\left(1+\left(\frac{\partial U}{\partial x}\right)^2\right)\frac{\partial^2 U}{\partial x^2} + \left(1+\left(\frac{\partial U}{\partial y}\right)^2\right)\frac{\partial^2 U}{\partial y^2} - 2\frac{\partial U}{\partial x}\frac{\partial U}{\partial y}\frac{\partial^2 U}{\partial x \partial y} = 0$$

$\nwarrow$ proportional to U^3

$\Rightarrow$ Nonlinear.

Prop 18.12 (Superposition) If $U_1, \dots U_n$ solve a linear PDE, then so does any linear combination

$$c_1 U_1 + \dots + c_n U_n.$$

Ex 18.13: Suppose U_1, U_2 solve Heat equation

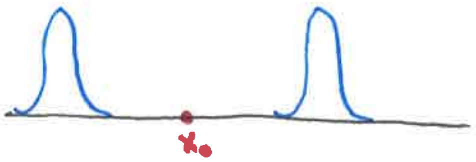
$$\partial U_1/\partial t = \alpha \frac{\partial^2 U_1}{\partial x^2}$$

$$\frac{\partial U_2}{\partial t} = \alpha \frac{\partial^2 U_2}{\partial x^2}$$

$$\frac{\partial}{\partial t}(aU_1 + bU_2) = a\frac{\partial U_1}{\partial t} + b\frac{\partial U_2}{\partial t}$$

$$= a\alpha\frac{\partial^2 U_1}{\partial x^2} + b\alpha\frac{\partial^2 U_2}{\partial x^2} = \alpha\left(\frac{\partial^2}{\partial x^2}(aU_1 + bU_2)\right). \checkmark$$

time $t=1$.



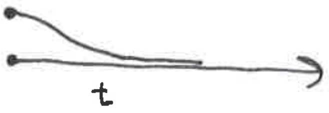
Superposition means that at x_0 heat experienced is just sum from two bumps, ie heat doesn't interact w/ itself.

Laser pulses

Superposition for wave equation means last pulses just pass through each other (but interfere when overlapping).

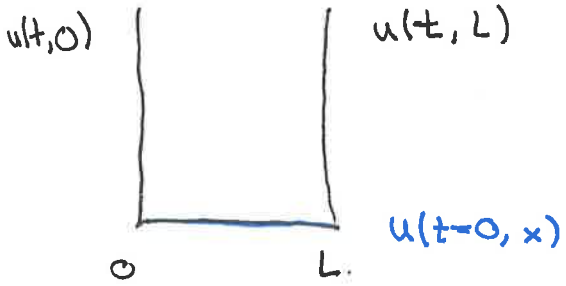
18.iii) Boundary-Value Problems

To solve an ODE on $[T_0, \infty)$



We had to specify n initial conditions at T_0, the boundary of domain, for an nth order equation.

Ex 18.14: Consider $U : [0,L] \times [0,\infty)$ ~~solving~~ and the heat equation



initial heat configuration

Def 18.15: The boundary Value Problem for the Heat/Wave equation is (to find) the solution of $\frac{\partial U}{\partial t} = \alpha \frac{\partial^2 U}{\partial x^2}$ (resp heat equation) with boundary specifications

- $U(0,x) = u_0(x)$ (initial condition)
- $U(t,0)$, $U(t,L) = v_1(t)$, $v_2(t)$ (often zero) Dirichlet (boundary condition)

OR

$\frac{\partial U}{\partial x}(t,0)$, $\frac{\partial U}{\partial x}(t,L) = v_1(t)$, $v_2(t)$ Neumann (boundary condition)

Lecture 19 | PDEs II : Separation of Variables

Recall the general strategy for ODEs

I) Apply a solving technique to get specific solutions
II) Take linear combinations to get general solution
III) Plug in boundary conditions to solve for ~~ab~~ constants

19.i) Separation of Variables : Consider the heat or wave equation on $\underset{x}{[0,L]} \times \underset{t}{[0,\infty)}$

Ansatz 19.1 : Assume that the solution has the form

$$U(t,x) = T(t)X(x).$$

ie is a product of functions of a single variable.

Derive an equation T, X satisfy:

$$\frac{\partial}{\partial t} T(t)X(x) = T'(t)X(x) \qquad\qquad T'(t)X(x) = T(t)X''(x)$$

$$\frac{\partial^2}{\partial x^2} T(t)X(x) = T(t)X''(t) \qquad\qquad \frac{T'(t)}{T(t)} = \frac{X''(x)}{X(x)}$$

Fact 19.2 : If $f(s) = g(u)$ for all s and all u, then both are constant.

$f(s) = g(0)$ for all s
$g(u) = f(\cancel{s}0) = g(0)$,

$$\Rightarrow \boxed{\begin{aligned} T'(t) &= \lambda T(t) \\ X''(x) &= \lambda X(x) \end{aligned}}$$

(reduced to two ODEs).
for some $\lambda \in \mathbb{R}$.

Ex 19.3 : Solve the X equation, with boundary conditions

$$X(0) = 0 \qquad X(L) = 0.$$

$$X(t) = C_1 \sin(\sqrt{\lambda}\, t) + C_2 \cos(\sqrt{\lambda}\, t).$$

$$0 = X(0) = C_2 \cdot 1 + C_1 \cdot 0 \Rightarrow C_2 = 0.$$

$$0 = X(L) = C_1 \sin(\sqrt{\lambda} \cdot L) = 0 \quad \text{so} \quad \sqrt{\lambda} L = \pi \cdot n \quad \text{for } n = 0, 1, -1, \ldots$$

$$\lambda L^2 = \pi^2 n^2$$

$$\lambda = \frac{\pi^2 n^2}{L^2}$$

(C_1 is arbitrary since linear).

$$X(x) = C\sin\left(\frac{n\pi}{L}x\right)$$

Ex 19.4 : (Solving the T equation).

Suppose that $X(0) =$ [illegible] $C_n \sin\left(\frac{n\pi}{L}x\right)$.

$$T'(t) = \lambda T(t) \qquad \Rightarrow \quad T(t) = e^{-\frac{n^2\pi^2}{L^2}t}. \qquad T(0)=1.$$
$$= \frac{n^2\pi^2}{L^2}T(t).$$

By the principle of superposition, we find:

Thm 19.5 : The solution of the boundary value problem

$$\begin{cases} \frac{\partial U}{\partial t} = \frac{\partial^2 U}{\partial x^2} \\ U(\overset{t,}{0}) = U(t,L) = 0 \\ U(0,x) = \sum_{n=1}^{N} C_n \sin\left(\frac{\pi n}{L}x\right) \end{cases}$$

is $$U = \sum_{n=1}^{N} C_n e^{-n^2\pi^2/L^2 \cdot t} \sin\left(\frac{\pi n}{L}x\right)$$

In fact, infinite sums also work!!

19.ii) General + Neumann boundary Conditions

Consider now the same but

$$X(0) = a \quad X(0) = b.$$

One could try to solve

$$X(t) = C_1\cos(\sqrt{\lambda}t) + C_2\sin(\sqrt{\lambda}t).$$

Or use the following strategy:

Ex 19.6 : $U_\infty(x) = a + \left(\frac{b-a}{L}\right)x$ is a solution of the heat equation.

• time ind, $\frac{\partial^2 U}{\partial x^2} = 0$. ✓.
$=0$.

By superposition $U(t,x) = U_\infty(x) + U(t,x)$ ↙ solves $\begin{cases} \text{heat equation} \\ \text{[illegible]}(0) = \text{[illegible]} \quad U(\overset{t,}{0}) = 0 \\ U\text{[illegible]}(0,x) = U_0. \quad U(t,L) = 0 \end{cases}$

Therefore $U = U_{ab}(x) + \sum_{n=1}^{\infty} c_n e^{-n^2\pi^2/L^2 x^2} \sin\left(\frac{n\pi}{L} x\right)$

solves

$$\begin{cases} \frac{\partial U}{\partial t} = \frac{\partial^2 U}{\partial x^2} \\ U(t,0) = a \\ U(t,L) = b \\ U(0,x) = U_{ab} + \sum c_n \sin\left(\frac{n\pi}{L} x\right). \end{cases}$$

Ex 19.7 : Find solution for $u(0) = u(L) = 0$

$u(0,x) = 4\sin\left(\frac{\pi}{\ell} x\right) - 5\sin\left(\frac{7\pi}{\ell} x\right).$

see figure.

Thm 19.8 :

$$\begin{cases} \frac{\partial U}{\partial t} = \frac{\partial^2 U}{\partial x^2} \\ X'(0) = X'(L) = 0 \\ U(0,x) = \;.. \end{cases}$$

Neumann boundary conditions are same except with cosine.

$$U = \sum_{n=\cancel{0}1}^{\infty} c_n e^{-n^2\pi^2/L^2 \cdot t} \cos\left(\frac{n\pi}{L} \cdot x\right).$$

Rem 19.9 : There is an analogy between decoupling and separation of variables.

$$X''(x) = \lambda X(x).$$

is an "eigenvector" of the equation/matrix $A = \frac{\partial^2}{\partial x^2}$.

19.iii) The Wave Equation:

The wave equation can be solved by separation of variables in the precisely analogous way.

Assume

$$U(t,x) = T(t) X(x).$$

Then

$$\frac{\partial^2}{\partial t^2} T(t) X(x) = T''(t) X(x)$$
$$c^2 \frac{\partial^2}{\partial x^2} T(x) X(x) = T(t)\, c^2 X''(x).$$

$$\Rightarrow \quad \frac{T''(t)}{T(t)} = \frac{c^2 X''(x)}{X(x)}. \quad \Rightarrow \begin{cases} T''(t) = \lambda T(t) \\ X''(x) = \lambda X(t) \end{cases}$$

$$T(t) = \cos(\sqrt{\lambda} c t) \qquad T(0) = 1.$$

For time equation.

$$X(x) = \sin\left(\frac{n\pi}{L} x\right). \qquad X(0) = X(L) = 0.$$

Thm 19.10 : The solutions of the BVP

$$\begin{cases} \frac{\partial^2 U}{\partial t^2} = c^2 \frac{\partial^2 U}{\partial x^2} \\ U(0) = U(L) = 0 \\ U(0,x) = \sum_1 C_n \sin\left(\frac{n\pi}{L} x\right) \end{cases}$$

is

$$U(t,x) = \sum_{n=0}^{\infty} C_n \sin\left(\frac{n\pi}{L} x\right) \cdot \cos\left(\frac{n\pi c}{L} t\right).$$

19.iv) Demonstration w/ wave equation

Lecture 20 Fourier Series I: Orthogonality + Coefficients

Recall that we solved the heat and wave equations for initial conditions

$$u(0,x) = \sum_{n=0}^{\infty} b_n \sin\left(\frac{n\pi}{L}x\right)$$

on $[0,L]$ (or cosine w/ Neumann boundary).

Question 20.1: What functions can be expressed as infinite sums of sin, cos?

The incredible answer (due to Fourier) is all of them!

He's french, the last R is silent
Fou - ee - ay

20.i) Periodic Functions

Def 20.2: A function $f: \mathbb{R} \to \mathbb{R}$ (or $\mathbb{C}$) is called L-periodic if

$$f(x+L) = f(x)$$

for all x.

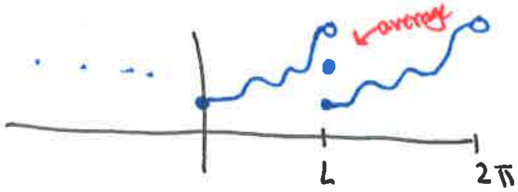
Ex 20.3: $$\sin\left(\frac{2n\pi}{L}x\right) = \sin\left(\frac{2n\pi}{L}(x+L)\right) = \sin\left(\frac{2n\pi x}{L} + 2n\pi\right)$$
$$= \sin\left(\frac{2n\pi x}{L}\right).$$

Note the 2!

is a sol of the ODE but is 4π periodic not 2π!

Same for cosine.

Ex 20.4: The functions need not be continuous



⇒ in this case take $f(0) = f(L)$ to be the average.

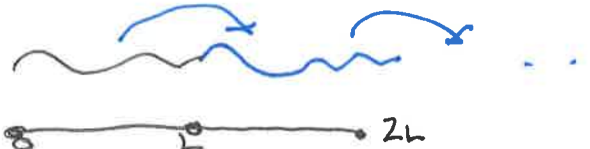
Prop 20.5: If $f(x)$ is L periodic, and $g(x)$ same

i) $f(x+a)$ for any a

ii) $f(nx)$ for $n = 1, 2, \ldots$ (in fact it's $\frac{L}{n}$ periodic)

iii) linear combinations $\alpha f(x) + \beta g(x)$.

Ex 20.5.5: Every function $f: [0,L]$ has a periodic extension by translation



20.ii) Fourier Series

Consider $\mathbb{R}^n$. It has a basis $e_1=\begin{pmatrix}1\\0\\\vdots\\0\end{pmatrix}$ $e_2=\begin{pmatrix}0\\1\\\vdots\\0\end{pmatrix}$ $\ldots$ $e_n=\begin{pmatrix}8\\\vdots\\1\end{pmatrix}$.

and a dot product $\langle v,w\rangle = v\cdot w = v_1w_1+\ldots+v_nw_n$.

Key Observation 20.6: Any vector $v=\begin{pmatrix}v_1\\\vdots\\v_n\end{pmatrix}$ can be written in the basis

$$v=\sum_{i=1}^{n} v_ie_i = v_1\begin{pmatrix}1\\0\\\vdots\\0\end{pmatrix}+v_2\begin{pmatrix}0\\1\\\vdots\\0\end{pmatrix}+\ldots+v_n\begin{pmatrix}8\\\vdots\\1\end{pmatrix}.$$

$$=\langle v,e_1\rangle\begin{pmatrix}1\\0\\\vdots\\0\end{pmatrix}+\langle v,e_2\rangle\begin{pmatrix}0\\1\\\vdots\\1\end{pmatrix}+\ldots\langle v,e_n\rangle\begin{pmatrix}8\\\vdots\\1\end{pmatrix}.$$

$$=\sum_{i=1}^{n}\langle v,e_i\rangle e_i$$

$\Rightarrow$ when writing vector in the basis, the coefficients are the dot product.

Note the basis satisfies: i) $\langle e_i,e_i\rangle=1$ $\leftarrow$ $\frac{1}{2}$ works just as well. basis is called orthonormal if it satisfies these.

ii) $\langle e_i,e_j\rangle=0$ $i\neq j$

iii) are linearly ind. and span.

Def 20.7: Given two functions $f,g:[0,L]\longrightarrow\mathbb{R}$, define the dot product by

$$\langle f,g\rangle=\frac{1}{L}\int_0^L f(x)g(x)dx.$$

Observe it satisfies

i) $\langle c_1f_1+c_2f_2,g\rangle=\frac{1}{L}\int_0^L\big(c_1f_1(x)+c_2f_2(x)\big)g(x)dx=c_1\langle f_1,g\rangle+c_2\langle f_2,g\rangle.$

ii) same for g

iii) $\langle f,g\rangle=\langle g,f\rangle$

Theorem 20.8 (Fourier theorem A). The functions

$$\frac{1}{2L},\ \sin\left(\frac{2\pi}{L}x\right),\ \cos\left(\frac{2\pi}{L}x\right),\ \sin\left(\frac{4\pi}{L}x\right),\ \overset{\cos}{\cancel{\sin}}\left(\frac{4\pi}{L}x\right),\ \ldots\ \begin{matrix}\sin(\frac{2\pi}{L}nx)\\ \cos(\frac{2\pi}{L}nx)\end{matrix}$$

are an orthonormal basis for L-periodic functions.

To see why, we just compute some Integrals, Let's take $L = 2\pi$.

Consider

- $$\langle \cos(nx), \cos(nx) \rangle = \frac{1}{2\pi}\int_0^{2\pi} \cos(nx)^2\,dx$$
$$= \frac{1}{2\pi}\int_0^{2\pi} \frac{1}{2} + \frac{1}{2}\cos(2nx)\,dx$$
$$= \frac{\pi}{2\pi} + \frac{1}{2\pi}\cancelto{0}{\left[\frac{1}{4\cdot n}\sin(2nx)\right]_0^{2\pi}} = \frac{1}{2}.$$

- $\langle \sin(nx), \sin(nx) \rangle = \frac{1}{2}$ same

- $m \neq n$
$$\langle \cos(nx), \sin(mx) \rangle =$$

$$\sin((n+m)x) = \sin(nx)\cos(mx) + \cos(nx)\sin(mx) \;..$$
$$\cos((n+m)x) = \;..$$
$$\frac{1}{2\pi}\int_0^{2\pi} \cos(nx)\sin(mx)\,dx = \frac{1}{2\pi}\int_0^{2\pi} \frac{1}{2}\sin((n+m)x) - \frac{1}{2}\sin((n-m)x)\,dx$$
$$= 0.$$

other cases same.

- linearly independent since none are multiples + span (this is trickier).

Theorem 20.9 (Fourier Series) If $f : [0, L] \to \mathbb{R}$ on L-periodic
$$f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos\left(\frac{2\pi n}{L}x\right) + \sum_{n=1}^{\infty} b_n \sin\left(\frac{2\pi n}{L}x\right).$$
where
$$a_0 = \langle f, 1 \rangle = \frac{1}{L}\int_0^L f(x)\,dx$$
$$a_n = \langle f, \cos(nx) \rangle = \frac{2}{L}\int_0^L f(x)\cos(nx)\,dx$$
$$b_n = \langle f, \sin(nx) \rangle = \frac{2}{L}\int_0^L f(x)\sin(nx)\,dx.$$

Ex 20.10: Compute the Fourier series of
$$f(x) = \begin{cases} 1 & 0 \le x \le \pi \\ 0 & \pi \le x \le 2\pi \end{cases}$$

$$a_0 = \frac{1}{2\pi}\int_0^{2\pi} f(x)\,dx = \frac{\pi}{2\pi} = \frac{1}{2}.$$
$$a_n = \frac{2}{2\pi}\int_0^{2\pi} f(x)\cos(nx)\,dx = \frac{1}{\pi}\int_0^{\pi} \cos(nx)\,dx = \frac{1}{\pi}\left.\frac{\sin(nx)}{n}\right|_{x=0}^{x=\pi} = \frac{1}{\pi}\left[\frac{\sin(n\pi) - 0}{n}\right] = 0.$$

$$b_n = \frac{2}{2\pi}\int_0^{2\pi} f(x)\sin(nx)dx = \frac{1}{\pi}\int_0^{\pi}\sin(nx)dx = \frac{1}{\pi}\left[\frac{-\cos(nx)}{n}\right]_0^{\pi}$$

$$= \frac{1}{\pi}\left[\frac{-\cos(n\pi)+1}{n}\right]$$

$$= \begin{cases} \frac{2}{n\pi} & (n \text{ odd}) \\ 0 & n \text{ even} \end{cases}$$

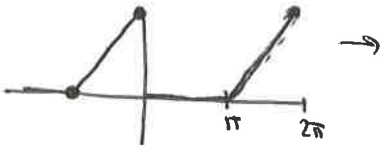
$$f(x) = \frac{1}{2} + \sum_{\substack{n \\ \text{odd}}}^{\infty} \frac{2}{n\pi}\sin(nx).$$

$\Rightarrow$ Show convergence graphic.

Shortcut 20.11: If you put f on $\underset{-L/2 \qquad\quad L/2}{\longmapsto}$ by translation it is even or odd sometimes $\Rightarrow$ sin, cos terms $=0$ respectively (even / odd up to subtracting a_0 works, as in previous example).

Ex 20.12: Let f(x) be the sawtooth

$$f(x) = \begin{cases} 0 & 0 \le x \le \pi \\ x-\pi & \pi \le x \le 2\pi \end{cases}$$



$$a_0 = \frac{1}{2\pi}\int_0^{2\pi} f(x)\,dx = \frac{1}{2\pi}\int_{\pi}^{2\pi}(x+\pi)dx = \frac{1}{2\pi}\int_0^{\pi} u\,du = \left.\frac{u^2}{4\pi}\right|_0^{\pi} = \frac{\pi^2}{4\pi} = \frac{\pi}{4}$$

$u = x+\pi.$

$$a_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\cos(nx)dx = \frac{1}{\pi}\int_{\pi}^{2\pi}(x-\pi)\cos(nx)dx = \frac{1}{\pi}\int_0^{\pi} u\cos(n(u+\pi))dx$$

$$b_n = \qquad \cdots \qquad = \frac{1}{\pi}\int_0^{\pi} u\sin(n(u+\pi))dx.$$

$\sin(nu+n\pi) = \sin(nu)$ n even
$\qquad\qquad\qquad -\sin(nu)$ n odd

$$b_n = \frac{(-1)^n}{\pi}\int_0^{\pi} u\sin(nu)du$$

$t = nu$ $\qquad u = t \quad dv = \sin t$

$$\int = \frac{1}{n^2}\int t\sin t\,dt = \frac{1}{n^2}\left[-t\cos t + \int \cos t\,dt\right) = \frac{1}{n^2}\left[-t\cos t + \sin t\right]$$

$$b_n = \frac{(-1)^n}{\pi n^2}\left[-t\cos t + \sin t\right]_{t=0}^{t=n\pi} = \frac{(-1)^n}{\pi n^2}\left(-n\pi\cos(n\pi)\right) = \frac{(-1)^n}{n}\left(-(-1)^n\right) = -\frac{1}{n}.$$

With more integration by parts

$$a_n = 0 \qquad n \text{ even}$$

$$a_n = \frac{2}{n^2\pi} \qquad n \text{ odd}$$

$$f(x) = \frac{\pi}{4} + \sum_{n \text{ odd}} \frac{2}{n^2\pi}\cos(nx) - \sum_{n=0}^{\infty} \frac{1}{n}\sin(nx).$$

<u>Theorem 20.13</u> : $\dfrac{\pi^2}{8} = 1 + \dfrac{1}{9} + \dfrac{1}{25} + \dfrac{1}{49} + \dots$

At $x = 0$ $f(x) = \frac{\pi}{2}$ (average) so

$$\frac{\pi}{2} = \frac{\pi}{4} + \sum_{n=1,3,5\dots} \frac{2}{n^2\pi} \Rightarrow \frac{\pi}{4} = \frac{2}{\pi}\left(1 + \frac{1}{9} + \frac{1}{25} + \dots\right)$$

Lecture 21 Fourier Series II : Solving PDEs.

Recall that $f : [0, L] \longrightarrow \mathbb{R}$ can be extended to an L-periodic function and

$$f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos\left(\frac{2\pi n}{L} x\right) + \sum_{n=1}^{\infty} b_n \sin\left(\frac{2\pi n}{L} x\right)$$

Def 21.1 : The sequences $\{a_n\}\{b_n\}$ are called the Fourier coefficients.

21.i) More on odd vs even

Recall that a function is odd if $f(-x) = -f(x)$
even if $f(-x) = f(x)$.

Since sin, cos are odd, even,

Prop 21.2 : If $f : [0, L] \longrightarrow \mathbb{R}$, and its periodic extension $f : [-\frac{L}{2}, \frac{L}{2}] \longrightarrow \mathbb{R}$ is

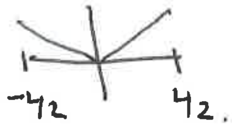
- Odd, then $a_n = 0$ for all n.
- Even, then $b_n = 0$ for all n.

$$a_n = \frac{2}{L} \cdot 2 \int_0^{L/2} f(x)\cos(nx)\,dx$$

$$b_n = \frac{2}{L} \cdot 2 \int_0^{L/2} f(x)\sin(nx)\,dx$$

Ex 21.3 : $f(x) = x$ has $a_n = 0$

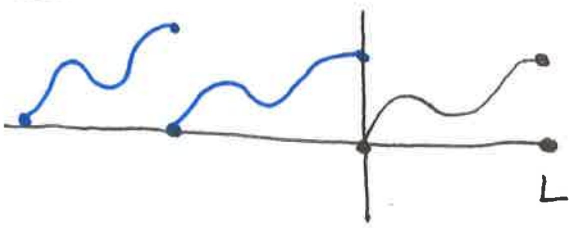
$f(x) = |x|$ has $b_n = 0$



⚠ This next part is quite confusing in the book. Let's explain step by step.

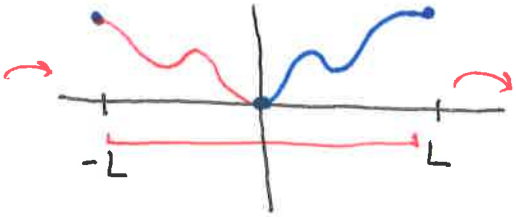
We are glossing over details on "extending" a function $f : [0, L] \longrightarrow \mathbb{R}$ to a periodic function. There are three ways to do this

(I)



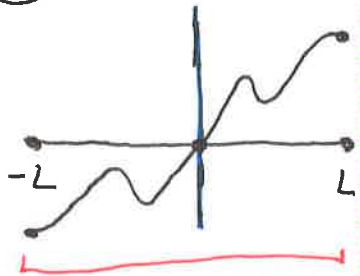
L-periodic extension

(II)



2L-periodic EVEN extension

(III)



2L-Periodic ODD extension.

Observe that extension

(I) is L-periodic, neither even nor odd ➔ $a_0 + \sum_{n=1}^{\infty} a_n \cos\left(\frac{2\pi n}{L}\cdot x\right) + \sum_{n=1}^{\infty} b_n \sin\left(\frac{2\pi n}{L}\cdot x\right)$.

(II) is $2L$-periodic and odd ➔ $\sum_{n=1}^{\infty} B_n \sin\left(\frac{2\pi n}{\boxed{2L}} x\right)$ Fourier Sine Series

*

(III) is $2L$-periodic and even ➔ $\sum_{n=0}^{\infty} A_n \cos\left(\frac{2\pi n}{\boxed{2L}} x\right)$ * Fourier Cosine Series

Observation 21.4: The extensions satisfy

(II) has $f(0) = f(L) = 0$ (Dirichlet)

(III) has $f'(0) = f'(L) = 0$ (Neumann)

because $f(L) = \sum B_n \sin\left(\frac{\pi n}{L}\cdot L\right) = 0$. for all n.

21.ii): Fourier Series and the Heat Equation

Recall we solve the Heat equation on $\{t \geq 0\} \times [0, L]$ provided that $U(0,x) = \sum (\text{sines})$

Because of the above, any function $U(0,x) = f(x)$ with $f(0) = f(L) = 0$ can be written this way via Fourier Sine series.

Thm 21.5 (Dirichlet Problem for the Heat Equation)

The initial value problem

$$\begin{cases} \frac{\partial U}{\partial t} = \alpha \frac{\partial^2 U}{\partial x^2} \\ U(0,x) = f(x) \end{cases}$$

with Dirichlet boundary condition $U(t,0) = U(t,L) = 0$ has (unique) solution

$$U(t,x) = \sum_{n=1}^{\infty} B_n \sin\left(\frac{2\pi n}{2L} x\right) \cdot e^{-\frac{\pi^2 n^2}{L^2} t}.$$

where

$$B_n = \frac{2}{L}\int_0^L f(x) \cdot \sin\left(\frac{2\pi n}{2L} \cdot x\right) dx.$$

More generally, for $U(t,0) = a$
$U(t,L) = b$

add $\left(a - \frac{b-a}{L}x\right)$ + same w/ $\tilde{B}_n$ defined by $\tilde{f}(x) = f(x) - \left(a - \frac{(b-a)}{L}x\right)$

Thm 21.6 (Neumann Problem for the heat equation)

For $U'(t,0) = 0$
$U'(t,L) = 0$

Everything is the same but w/ Fourier cosine series.

$$U(t,x) = a_0 + \sum_{n=1}^{\infty} A_n \cos\left(\frac{2\pi n}{2L}x\right) e^{-\frac{\pi^2 n^2}{L^2}t}$$

Rem 21.7: The interpretation of the boundary conditions is

Dirichlet = ends are held at constant temperature

Neumann = ends are perfectly insulted so heat flux in/out is zero.

21.iii) Fourier Series and the Wave Equation

The time equation $T''(t) = \lambda T(t)$ is now second order, so we must prescribe both $U(0,x)$ and $\partial_t U(0,x)$.

Thm 21.8 (Dirichlet Problem for the Wave Equation)

The IVP

$$\begin{cases} \frac{\partial^2 U(t,x)}{\partial t^2} = c^2 \frac{\partial^2 U(t,x)}{\partial x^2} \\ U(0,x) = f(x) \\ \frac{\partial}{\partial t} U(0,x) = g(x) \end{cases}$$

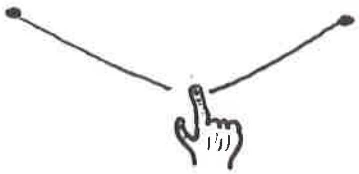
where $A_n = \frac{2}{L}\int_0^L f(x) \sin\left(\frac{2\pi n}{2L}x\right) dx$

$B_n = \frac{2}{L}\int_0^L g(x)$

with Dirichlet boundary conditions $U(t,0) = U(t,L) = 0$ is

$$U(t,x) = \underbrace{\sum_{n=1}^{\infty}\left[A_n \cos\left(\frac{2n\pi}{2L}t\cdot c\right) + B_n \sin\left(\frac{2n\pi}{2L}ct\right)\right]}_{T(t)} \cdot \underbrace{\sin\left(\frac{2n\pi}{2L}x\right)}_{X(x)}.$$

Ex 21.9 (Plucked String)



$$U(0,x) = -\frac{a}{L/2}\left|x - \frac{L}{2}\right| + a$$

$$\partial_t U(0,x) = 0 .$$

$$A_0 = \frac{1}{L}\int_0^L f(x)\,dx = \frac{1}{L}\cdot\frac{L}{2}\cdot a = \frac{a}{2}$$

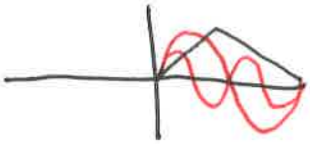
$$A_n = \frac{2}{L}\int_0^L f(x)\sin\left(\frac{2n\pi}{2L}x\right)$$

$$= \frac{4}{L}\int_0^{L/2} -\frac{a}{L/2}x\ \sin\left(\frac{\pi n}{L}x\right)dx$$

$$= \frac{-4a}{L^2}\cdot 2\int_0^{L/2} x\ \sin(\quad)dx \qquad y = x\frac{\pi}{L}$$

$$= \frac{-8a}{L\pi}\int_0^{\pi/2} x\sin(nx)\,dx$$

$$= \frac{-8a}{L\pi}\left[\frac{(-1)^{\frac{n-1}{2}}}{n^2}\right] \qquad n \text{ odd}.$$



n = even cancels

⇒ Video.

Rem 21.10: Note the difference between the two equations

$e^{-\frac{n^2\pi^2}{L^2}t}\sin(x)$ dampening to zero exponentially quickly

⇒ constant linear temp equilibrium of

$U_\infty = a - \frac{(b-a)}{L}x$ as $t \to \infty$

approaches this exponentially w/ exponent $e^{-\frac{\pi^2}{L^2}}$.

$\cos\left(\frac{n\pi}{L}ct\right)\sin\left(\frac{n\pi}{L}x\right)$ wave equation has no dampening w/o extra friction terms.

Lecture 22 | Fourier Series III: higher-dimensional examples.

Recall that on $[0,L]$ the heat and wave equations were solved by separation of variables. Here we will show the same technique works in more dimensions.

22.i) The Heat Equation on the Square

Recall the heat equation (for $\alpha = 1$) in two (spatial) dimensions is

$$\frac{\partial U}{\partial t} = \frac{\partial^2 U}{\partial x^2} + \frac{\partial^2 U}{\partial y^2}$$

Notation 22.1: The sum of second derivatives

$$\begin{matrix}\Delta U \\ \text{or } \nabla^2 U\end{matrix} = \frac{\partial^2 U}{\partial x^2} + \frac{\partial^2 U}{\partial y^2}$$

is called the Laplacian.

Consider the rectangle $\Omega = [0, L_1] \times [0, L_2]$.

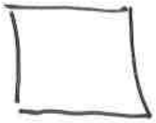
$$= \{(x,y) \in \mathbb{R}^2 \mid 0 \le x \le L_1 ,\ 0 \le y \le L_2\}.$$

Def 22.2: The initial value problem for the heat equation w/ Dirichlet boundary conditions is

$$\begin{cases} \frac{\partial U}{\partial t} = \nabla^2 U \\ U(0,x,y) = g(x,y) \end{cases}$$

and $U \equiv 0$ on $\partial\Omega$,* ie

$$0 = U(t,0,y) = U(t,L_1,y) = U(t,x,0) = U(t,x,L_2).$$



Ansatz 22.3: Suppose that

$$U(t,x,y) = T(t)X(x)Y(y).$$

Then

$$T''(t)X(x)Y(y) = T(t)X''(x)Y(y) + T(t)X(x)Y''(y)$$

$$\frac{T''(t)}{T(t)} = \lambda = \frac{X''(x)}{X(x)} + \frac{Y''(y)}{Y(y)}.$$

By the same logic as before,

$$\frac{X''(x)}{X(x)} = \lambda - \underbrace{\frac{Y''(y)}{Y(y)}}_{=\lambda_1} \qquad \text{for all } x \Rightarrow \text{const.}$$

$$\frac{X''(x)}{X(x)} = \lambda_1 \qquad \frac{Y''(y)}{Y(y)} = \lambda_2 .$$

The solutions are the same as before in each separate direction.

Thm 22.4: The solution (unique) of the IVP

$$\begin{cases} \frac{\partial U}{\partial t} = \nabla^2 U \\ U(0,x,y) = g(x,y) \end{cases}$$

w/ Dirichlet boundary conditions

$$U(t,x,y) = \sum_{j=1}^{\infty}\sum_{k=1}^{\infty} B_{jk}\, e^{-\left(\frac{j^2}{L_1^2}+\frac{k^2}{L_2^2}\right)\pi^2 t} \sin\left(\frac{j\pi}{L_1}x\right)\sin\left(\frac{k\pi}{L_2}y\right).$$

where B_{jk} are the two-dimensional Fourier coefficients

$$B_{jk} = \frac{4}{L_1L_2}\int_0^{L_2}\int_0^{L_1} g(x,y)\sin\left(\frac{j\pi}{L_1}x\right)\sin\left(\frac{k\pi}{L_2}y\right)dx\,dy .$$

Ex 22.5: If $g(x,y) = 8\sin(3x)\sin(7y)$ then on $[0,1]\times[0,1]$

$$B_{jk} = \begin{cases} 8 & j,k = 3,7 \\ 0 & \text{else} \end{cases} \qquad \iint dx\,dy = \Big(\underbrace{\int dx}_{\substack{=1 \text{ if } j=3 \\ 0 \text{ else}}}\Big)\Big(\underbrace{\int dy}_{\substack{=1 \text{ if } k=7 \\ 0 \text{ else}}}\Big)$$

so $U(t,x,y) = 8e^{-(3^2+7^2)t}\sin(3x)\sin(7y)$.

If $g(x,y) = 5\sin(2x)\sin(3y) - 9\sin(x)\sin(2y)$

$$\left\{ B_{jk} = \begin{matrix} 5 & (2,3) \\ 9 & (1,2) \end{matrix} \right. \Rightarrow U = 5e^{-(2^2+3^2)t}\sin(2x)\sin(3y) - 9e^{-5t}\sin(x)\sin(2y).$$

22.ii) The Wave Equation on the disk

Consider the disk $D = \{(x,y) \mid x^2+y^2 \leq 1\} \subseteq \mathbb{R}^2$.

And the Wave equation

$$\begin{cases} \frac{\partial^2 U}{\partial t^2} = \nabla^2 U \\ U(0,x,y) = g(x,y) \\ \partial_t U(0,x,y) = h(x,y) \end{cases} \qquad U(t,x,y) = 0 \text{ on } x^2+y^2 = 1$$

Observation 22.6: Separation of variables does not work in x and y. The problem has spherical symmetry so let's change to polar.

Ansatz 22.7: Assume

$$U(t,r,\theta) = T(t)\,R(r)\,\Theta(\theta).$$

In polar coordinates the equation is

$$\frac{\partial^2 U}{\partial t^2} = \left[\frac{\partial^2 U}{\partial r^2} + \frac{1}{r}\frac{\partial U}{\partial r} + \frac{1}{r^2}\frac{\partial^2 U}{\partial \theta^2}\right].$$

$$\Rightarrow \quad T''(t)R(r)\Theta(\theta) = \left[R''(r)\Theta(\theta) + \frac{1}{r}R'(r)\Theta(\theta) + \frac{1}{r^2}\Theta''(\theta)R(r)\right]T(t).$$

$$\frac{T''(t)}{T(t)} = \lambda = \frac{R''(r) + \frac{1}{r}R'(r)}{R(r)} + \frac{1}{r^2}\frac{\Theta''(\theta)}{\Theta(\theta)}.$$

$$\Rightarrow \begin{cases} T''(t) = \lambda T(t). \quad \swarrow \text{ Seen that before } \checkmark. \\ r^2R''(r)\Theta(\theta) + rR'(r)\Theta(\theta) + \Theta''(\theta)R(r) + \lambda r^2 R(r)\Theta(\theta) = 0. \end{cases}$$

Divide by $\Theta \cdot R$

$$\frac{r^2R''(r) + rR'(r) + \lambda r^2R(r)}{R(r)} = \frac{\Theta''(\theta)}{\Theta(\theta)} = \mu.$$

$$\Rightarrow \begin{cases} T''(t) = \lambda T(t) & \text{seen before} \\ \Theta''(t) = \mu\Theta(\theta) & \text{same} \\ r^2R''(r) + rR'(r) + (\lambda r^2 + \mu)R(r) = 0. & \text{Bessel Equation.} \end{cases}$$

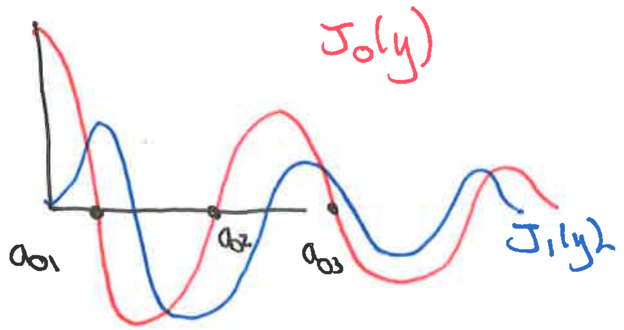
1) $\Theta''(\theta) = \mu\Theta(\theta)$ ⇒ $\mu = n^2$ $\Theta(\theta) = A_n\cos(n\theta) + B_n\sin(n\theta)$.

2) $y = \sqrt{\lambda}\, r$

$y^2R''(y) + yR'(y) + (y^2+n^2)R(y) = 0$ ⇒ Solution is

$J_n(y)$, $\cancel{Y_n(y)}$ Asyptote at 0

$J_n(\sqrt{\lambda}\, r)$



⇒ $J_n(y) = 0$ at a_{mn}.

The boundary condition holds if $\sqrt{\lambda}\cdot 1 = a_{mn}$ for some m,n.

$\lambda = a_{mn}^2$.

Putting it together

$$* \quad U(t,r,\theta) = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} \cos(\lambda_{mn} t)\, J_n(\sqrt{\lambda_{mn}}\, r)\left[A_n\cos(n\theta) + B_n\sin(n\theta)\right].$$

Prop 22.8: The general solution of wave eq on disk with Dirichlet boundary is

$U(t,r,\theta) = *$.

Rem 22.9: Display graphics.

Rem 22.10: Key Takeaways

1) Separation of variables is useful lots of places, but adapting it to each situation requires some thinking (here, polar separation natural b/c of geometry).

2) Special functions are real and do (annoyingly) appear in real problems.

Question 22.11: The λ_{mn} determine the vibrational frequencies, but are hard to calculate. They depend only on the circular geometry.

Q (Katz): Can you hear the shape of a drum? ie from measuring λ_{mn}, can you reconstruct the geometric domain (when not a disk)?

Lecture 23 Complex Fourier Series + Transforms.

Recall

$$e^{inx} = \cos(nx) + i\sin(nx).$$

It is often much easier to work with $\mathbb{C}$-valued functions.

Motivation 23.1

i) Many computations are much simpler w/ $\mathbb{C}$-valued functions.

$$e^{i(n+m)x} = e^{inx}e^{imx}$$

$\cos((n+m)x) =$ Idk, I don't remember either.

ib) Fourier series in particular is much neater over $\mathbb{C}$.

ii) Many applications (e.g. Schrödinger's Equation) naturally involve complex numbers.

23.i) Linear Algebra with $\mathbb{C}$

Def 23.2: A complex vector space with basis $v_1 \cdots v_n$ is

$$V = \mathbb{C}^n = \{\alpha_1 v_1 + \cdots + \alpha_n v_n \mid \alpha_j \in \mathbb{C}\}.$$

Def 23.3: The *Hermitian* or Complex inner product is $v = \sum \alpha_j v_j$ $w = \sum \beta_j v_j$

$$\langle v,w\rangle = \alpha_1\overline{\beta}_1 + \alpha_2\overline{\beta}_2 + \cdots + \alpha_n\overline{\beta}_n \in \mathbb{C}.$$

⚠ Note the complex conjugation. This ensures

$$\langle v,v\rangle = \alpha_1\overline{\alpha}_1 + \alpha_2\overline{\alpha}_2 + \cdots + \alpha_n\overline{\alpha}_n = \sum |\alpha_n|^2 > 0$$ is a Real Number

$\Rightarrow$ It makes sense to talk about length of vectors still (and $\mathbb{C}^n = \mathbb{R}^{2n}$ this is the same old length from Pythagoras).

Prop 23.4 (Properties of the Hermitian inner product

i) $\langle v,v\rangle = \|v\|^2 \geq 0$ is real

ii) $\langle v,w\rangle = \overline{\langle w,v\rangle}$

iii) $\langle v_1+v_2, w\rangle = \langle v_1,w\rangle + \langle v_2,w\rangle$

iv) $\langle \alpha v, w\rangle = \alpha\langle v,w\rangle$ but $\langle v,\alpha w\rangle = \overline{\alpha}\langle v,w\rangle$.

v) If $e_i = \begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix}$ ith spot, $v = \sum_{i=1}^{n} \langle v, e_i\rangle e_i$.

23.ii) Complex Fourier Series

There is also an inner product for complex periodic functions $f,g : [0,L] \to \mathbb{C}$.

$$\langle f,g\rangle_{\mathbb{C}} = \frac{1}{L}\int_0^L f(x)\overline{g(x)}\,dx$$

➡ i) - iv) still hold just like for $\mathbb{C}^n$.

Prop 23.5: The functions

! now include negatives!

$$e^{inx\cdot w} = \cos(nwx) + i\sin(nwx)$$

for $w = \frac{2\pi}{L}$ are an orthonormal basis for $n = \dots -2, -1, 0, 1, 2 \dots$ ✓

e.g.

$$\langle e^{inx}, e^{imx}\rangle = \frac{1}{L}\int_0^L e^{inwx}\overset{\text{conjugate!}}{e^{-imwx}}\,dx = \frac{1}{L}\int_0^L \cos((n-m)wx) + i\sin((n-m)wx))\,dx = 0$$

unless $m = n$.

Thm 23.6 (Complex Fourier Series) Every L-periodic $f : \mathbb{R} \to \mathbb{C}$ or $f : [0,L] \to \mathbb{C}$

can be written

$$f(x) = \sum_{n=-\infty}^{\infty} C_n e^{inwx}$$

when

$$C_n = \langle f, e^{inwx}\rangle = \frac{1}{L}\int_0^L f(x)e^{-inwx}\,dx = \frac{1}{L}\int_{-L/2}^{L/2} f(x)e^{-inwx}\,dx.$$

In this case $\lim_{N\to\infty} \sum_{n=-N}^{N} C_n e^{inwx} \to f(x)$ everywhere (average at discontinuities)

Rem 23.7 (Comparison w/ real Fourier series) Write $C_n = \alpha_n + i\beta_n$

And suppose $f : [0,L] \to \mathbb{R} \subseteq \mathbb{C}$ is secretly real-valued. Then

$$C_n = \frac{1}{L}\int_0^L f(x)e^{-inwx}\,dx = \frac{1}{L}\int_0^L f(x)e^{+inwx}\,dx = \overline{C_n}.$$

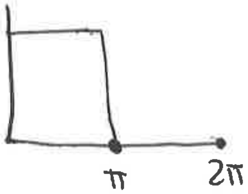
So

$$\sum_{n=-\infty}^{\infty} C_n e^{inwx} = \sum_{n=0}^{\infty} (\alpha_n + i\beta_n)(\cos(wnx) + i\sin(nwx)) + (\alpha_n - i\beta_n)(\cos(nwx) - i\sin(nwx))$$

$$= \sum_{n=0}^{\infty} 2\alpha_n\cos(wnx) - 2i\beta_n\sin(nwx)$$

so $a_n = 2\alpha_n$, $b_n = -2\beta_n$ reproduces real Fourier series.

Ex 23.8 :



$$f(x) = \begin{cases} 1 & 0 \le x \le \pi \\ 0 & \pi \le x \le 2\pi. \end{cases}$$

Since this is real, could use previous example. Or

$$C_n = \frac{1}{2\pi}\int_0^{2\pi} f(x)e^{-inx}\,dx = \frac{1}{2\pi}\int_0^{\pi} e^{-inx}\,dx = \frac{1}{2\pi}\left.\frac{e^{-inx}}{-in}\right|_{x=0}^{\pi}$$

$$= \frac{1}{2\pi}\left[\frac{(-1)^n-1}{-in}\right] = \begin{cases} \frac{1}{2\pi i n} & n \text{ odd} \\ 0 & n \text{ even} \ne 0. \end{cases}$$

$$C_0 = \frac{1}{2\pi}\cdot 1\cdot \pi = \frac{1}{2}.$$

$$\Rightarrow \quad f(x) = \frac{1}{2} + \sum_{\substack{n \text{ odd} \\ n=-\infty}}^{\infty} e^{inx}\cdot\frac{1}{\pi i n} = \frac{1}{2} + \sum_{\substack{n=1 \\ n \text{ odd}}}^{\infty} \frac{2\sin(nx)}{\pi n}.$$

Rem 23.9 : Notice that Complex Fourier Series refers to function

$$f : [0,L] \to \mathbb{C}$$

and not (double periodic) $f : \mathbb{C} \to \mathbb{C}$. The latter requires 2-dim Fourier series like the heat eq on the square. It's very useful in solid-state physics for studying lattice structures and leads to the theory on bands / conduction.

23.iii) Fourier Series as a transform

Recall that a function

$$\begin{aligned} f : [0,L] &\longrightarrow \mathbb{C} \\ x &\longmapsto f(x) \end{aligned}$$

is an operation that "eats" a number and returns a number. Contrast this with Fourier Series

$$\left\{ \begin{array}{c} \text{Complex valued} \\ \text{periodic functions} \\ f : [0,L] \to \mathbb{C} \end{array} \right\} \longrightarrow \left\{ \begin{array}{c} \text{Infinite sequences} \\ \{C_n\} \text{ of Fourier} \\ \text{coefficients} \end{array} \right\}$$

$$f(x) \longmapsto \{C_n\}_{n=-\infty}^{\infty}$$

Def 23.10 : A transform is an operation that takes as input a function or sequence and returns another function or sequence.

"a function from functions to functions" (or sequences)

* Observe that a sequence is just a function $F: \mathbb{Z} \to \mathbb{C}$ $\quad \mathbb{Z} = \ldots -1, 0, 1, 2, \ldots$

Ex 23.11 : Fourier series gives a transform

*

$$\left\{ \begin{matrix} \mathbb{C}\text{-valued periodic} \\ \text{functions} \end{matrix} \right\} \xrightarrow{\mathcal{F}} \left(\begin{matrix} \mathbb{C}\text{-valued sequences} \\ = \mathbb{C}\text{-valued functions on } \mathbb{Z} \end{matrix} \right).$$

$$f(x) \longmapsto \hat{f}(n) : \mathbb{Z} \to \mathbb{C}$$

$$\hat{f}(n) := c_n \quad \text{Fourier coefficients.}$$

* Sequence not series! Right side stores the "data" of the coefficients".

And taking the series gives a reverse transform

$$\left\{ \mathbb{C}\text{-value sequences} \right\} \xrightarrow{\mathcal{F}^{-1}} \left\{ \begin{matrix} \mathbb{C}\text{-valued periodic} \\ \text{functions} \end{matrix} \right\}$$

**

$$\hat{g}(n) \longmapsto \sum_{n=-\infty}^{\infty} \hat{g}(n) e^{inwx}$$

reconstruct function from coefficient data.

Def 23.12 * is called the Fourier Transform (for periodic functions)
** is called the Inverse Fourier Transform (")

The transform perspective reveals a new relationship between f and Fourier Coefficients

Thm 23.13 : For continuous functions the operations are inverses,

$$\mathcal{F}^{-1}\mathcal{F}(f) \overset{\text{i.e.}}{=} \sum_{n=-\infty}^{\infty} \hat{f}(n) e^{inwx} = f(x)$$

$$\mathcal{F}\mathcal{F}^{-1}(\hat{f}) = \left\{ \frac{1}{L} \int_0^L \left(\sum_{k=-\infty}^{\infty} \hat{f}(k) e^{ikwx} \right) e^{-inwx} \right\}_{n=-\infty}^{\infty} = \hat{f}(n).$$

And

$$\frac{1}{L} \int_0^L |f(x)|^2 dx = \sum_{n=-\infty}^{\infty} |\hat{f}(n)|.$$

(Parseval's Identity).

23.ii) Complex Fourier Series

There is also an inner product for complex periodic functions $f,g:[0,L]\to\mathbb{C}$.

$$\langle f,g\rangle_{\mathbb{C}} = \frac{1}{L}\int_0^L f(x)\overline{g(x)}\,dx$$

➡ i) - iv) still hold just like for $\mathbb{C}^n$.

Prop 23.5: The functions

$$e^{inx\cdot w} = \cos(nwx)+i\sin(nwx)$$

! now include negatives

✓

for $w=\frac{2\pi}{L}$ are an orthonormal basis for $n=\dots -2,-1,0,1,2\dots$

e.g.

$$\langle e^{inx}, e^{imx}\rangle = \frac{1}{L}\int_0^L e^{inwx}\overset{\text{conjugate!}}{e^{-imwx}}\,dx = \frac{1}{L}\int_0^L \cos((n-m)wx)+i\sin((n-m)wx))\,dx = 0$$

unless $m=n$.

Thm 23.6 (Complex Fourier Series) Every L-periodic $f:\mathbb{R}\to\mathbb{C}$ or $f:[0,L]\to\mathbb{C}$

can be written

$$f(x)=\sum_{n=-\infty}^{\infty} C_n e^{inwx}$$

when

$$C_n = \langle f, e^{inwx}\rangle = \frac{1}{L}\int_0^L f(x)e^{-inwx}\,dx = \frac{1}{L}\int_{-L/2}^{L/2} f(x)e^{-inwx}\,dx.$$

In this case $\lim_{N\to\infty}\sum_{n=-N}^{N} C_n e^{inwx}\to f(x)$ everywhere (average at discontinuities)

Rem 23.7 (Comparison w/ real Fourier series) Write $C_n=\alpha_n+i\beta_n$

And suppose $f:[0,L]\to\mathbb{R}\subseteq\mathbb{C}$ is secretly real-valued. Then

$$C_n = \frac{1}{L}\int_0^L f(x)e^{-inwx}\,dx = \frac{1}{L}\int_0^L f(x)e^{+inwx}\,dx = \overline{C_n}.$$

So

$$\sum_{n=-\infty}^{\infty} C_n e^{inwx} = \sum_{n=0}^{\infty}(\alpha_n+i\beta_n)(\cos(wnx)+i\sin(nwx)) + (\alpha_n-i\beta_n)(\cos(nwx)-i\sin(nwx))$$

$$=\sum_{n=0}^{\infty} 2\alpha_n\cos(wnx) - 2i\beta_n\sin(nwx)$$

so $a_n = 2\alpha_n$, $b_n = -2\beta_n$ reproduces real Fourier series.

Def 23.10: A **transform** is an operation that takes as input a function or sequence and returns another function or sequence.

"a function from functions to functions"
(or sequences)

* Observe that a sequence is just a function $F: \mathbb{Z} \to \mathbb{C}$ $\mathbb{Z} = \ldots -1, 0, 1, 2, \ldots$

Ex 23.11: Fourier series gives a transform

* $$\{\mathbb{C}\text{-valued periodic functions}\} \xrightarrow{\mathcal{F}} \{\mathbb{C}\text{-valued sequences} = \mathbb{C}\text{-valued functions on } \mathbb{Z}\}.$$

$$f(x) \longmapsto \hat{f}(n) : \mathbb{Z} \to \mathbb{C}$$

$$\hat{f}(n) := c_n \quad \text{Fourier coefficients.}$$

* Sequence not series! Right side stores the "data" of the coefficients",

And taking the series gives a reverse transform

** $$\{\mathbb{C}\text{-value Sequences}\} \xrightarrow{\mathcal{F}^{-1}} \{\mathbb{C}\text{-valued periodic functions}\}$$

$$\hat{g}(n) \longmapsto \sum_{n=-\infty}^{\infty} \hat{g}(n) e^{inwx}$$

reconstruct function from coefficient data.

Def 23.12 * is called the Fourier Transform (for periodic functions)
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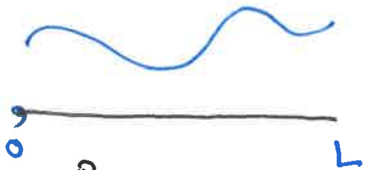
$$\mathcal{F}\mathcal{F}^{-1}(\hat{f}) = \left\{ \frac{1}{L}\int_0^L \left(\sum_{k=-\infty}^{\infty} \hat{f}(k) e^{ikwx} \right) e^{-inwx} \right\}_{n=-\infty}^{\infty} = \hat{f}(n).$$

And

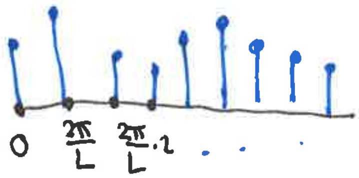
$$\frac{1}{L}\int_0^L |f(x)|^2 dx = \sum_{n=-\infty}^{\infty} |\hat{f}(n)|. \quad \text{(Parseval's Identity).}$$

Lecture 24 Fourier Transform I: Definition and Examples

Fourier series applies only to periodic functions, what about non-periodic functions?



Function on $[0,L]$



Series on $\frac{2\pi}{L}\mathbb{Z} \subseteq \mathbb{R}$.

Idea 24.1: For non-periodic functions take the limit as $L \longrightarrow \infty$.

The spacing of the series $\frac{2\pi}{L} \to 0$.

$\Rightarrow$ Series becomes a function

$$\hat{f} : \mathbb{R} \to \mathbb{C}$$

also.

Def 24.2: The Fourier transform of a (not necessarily periodic function $f : \mathbb{R} \to \mathbb{C}$ is

$$\hat{f}(\lambda) := \mathcal{F}(f) = \int_{-\infty}^{\infty} f(x) e^{-i\lambda x}\, dx$$

* Integrating dx leaves dependence on λ.

and the inverse Fourier transform

$$g(x) = \mathcal{F}^{-1}(\hat{g}(\lambda)) := \frac{1}{2\pi}\int_{-\infty}^{\infty} \hat{g}(\lambda) e^{i\lambda x}\, d\lambda$$

$d\lambda$ . . x.

Thm 24.3 (Fourier-Parseval)

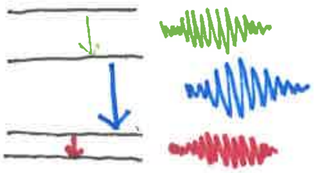
$$\mathcal{F}\mathcal{F}^{-1}(\hat{g}) = \hat{g} \quad \text{and} \quad \mathcal{F}^{-1}\mathcal{F}(f) = f$$

$$\int_{-\infty}^{\infty} |f(x)|^2 dx = \frac{1}{2\pi}\int_{-\infty}^{\infty} |\hat{f}(\lambda)|^2 d\lambda$$

are inverse transforms.

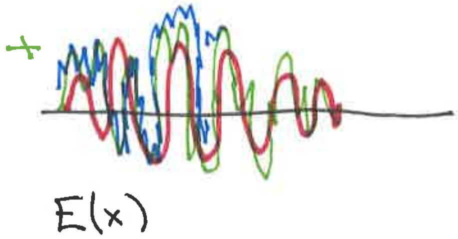
24.i) Intuition and Applications of the Fourier transform

Ex 24.4: Recall that atomic transitions produce light of certain frequencies



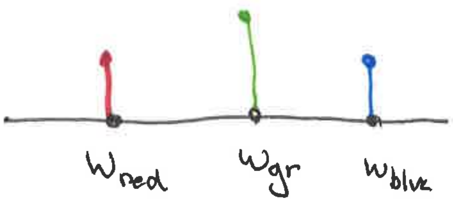
Emission light
from three
transitions
gives electric field

$$E(x) = A_1 \sin(\omega_{red}\, x) + A_2 \sin(\omega_{gr}\, x) + A_3 \sin(\omega_{blue}\, x)$$



$\Rightarrow$ Fourier transform.

$\hat{E}(\lambda)$.



The Fourier transform of the electric field waves is the frequency spectrum. The electric field describes the physics (wave eq) but the frequency spectrum is much more digestable so signal processing uses this. The Fourier transform is the dictionary for converting.

Ex 24.5 : To summarize

$$E(x) \xrightarrow{\mathcal{F}} \hat{E}(\omega)$$ (spatial profile to frequency profile)

$$\psi(x) \xrightarrow{\mathcal{F}} \hat{\psi}(p)$$ (position profile to momentum profile in QM).

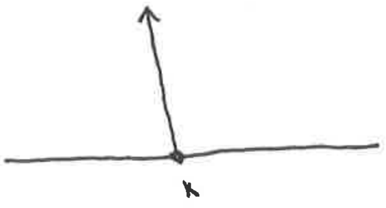
$$f(x) \xrightarrow{\mathcal{F}} \hat{f}(\lambda)$$ (function to its eigenvalue decomposition).

24.ii) Examples of Fourier transform pairs

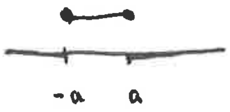
Ex 24.6 : $f(x) = \overset{= e^{ikx}}{\underset{\cos}{\cancel{\sin}}(kx)}$ then

$$\hat{f}(\lambda) = \int_{-\infty}^{\infty} e^{ikx} e^{-i\lambda x}\, dx = \begin{cases} 0 & \text{if } k \neq \lambda \\ \infty & \text{if } k = \lambda \end{cases}$$

$\Rightarrow$



Ex 24.7 : $f(x) =$

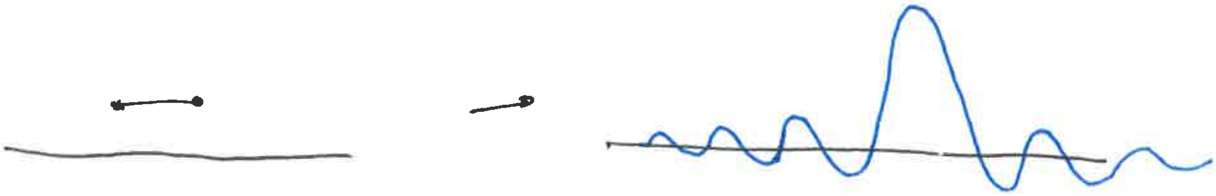


$$= \begin{cases} 1 & |x| \leq a \\ 0 & \text{else} \end{cases}$$

$$\hat{f}(\lambda) = \int_{-\infty}^{\infty} f(x) e^{-i\lambda x}\, dx = \int_{-a}^{a} e^{-i\lambda x} = \frac{1}{-i\lambda}\Big[e^{-i\lambda x} \Big]_{x=-a}^{x=a}$$

$$= \frac{e^{i\lambda a} - e^{-i\lambda a}}{i\lambda} = \frac{2\sin(\lambda a)}{\lambda} \qquad \lambda \neq 0.$$

$$\hat{f}(0) = 2a$$



Ex 24.8 : Let $f(x) = \begin{cases} 1 - \frac{|x|}{a} & |x| \leq a \\ 0 & |x| > a. \end{cases}$

$$\hat{f}(\lambda) = \int_{-\infty}^{\infty} f(x) e^{-i\lambda x} dx = \int_{-a}^{a} \left(1 - \frac{|x|}{a}\right) e^{-i\lambda x} dx$$

$$= \int_{-a}^{0} \left(1 + \frac{x}{a}\right) e^{-i\lambda x} dx + \int_{0}^{a} \left(1 - \frac{x}{a}\right) e^{-i\lambda x} dx$$

$$= \int_{-a}^{a} e^{-i\lambda x} dx + \frac{1}{a} \int_{-a}^{0} x e^{-i\lambda x} dx - \frac{1}{a} \int_{0}^{a} x e^{-i\lambda x}$$

$$= \frac{e^{-i\lambda x}}{-i\lambda} \Bigg|_{x=-a}^{x=a} + \frac{1}{a} \left[\frac{x e^{-i\lambda x}}{-i\lambda} + \frac{e^{-i\lambda x}}{\lambda^2} \right]_{x=-a}^{0} - \frac{1}{a} \left[\frac{x e^{-i\lambda x}}{-i\lambda} + \frac{e^{-i\lambda x}}{\lambda^2} \right]_{0}$$

$$= \frac{e^{i\lambda a} - e^{-i\lambda a}}{ia} + \frac{1}{a} \left(\frac{1}{\lambda^2} - \frac{a e^{i\lambda a}}{i\lambda} - \frac{e^{i\lambda a}}{\lambda^2} \right) - \frac{1}{a} \left(\frac{-a a^{-i\lambda a}}{i\lambda} + \frac{e^{-i\lambda a}}{\lambda^2} - \frac{1}{\lambda^2} \right)$$

$$= \frac{2 - (e^{i\lambda a} + e^{i\lambda a})}{a\lambda^2} = \frac{2(1 - \cos(\lambda a))}{a\lambda^2}$$

$\hat{f}(0) = a.$

$$= a \left(\frac{\sin(\lambda a/2)}{\lambda a/2} \right)^2$$

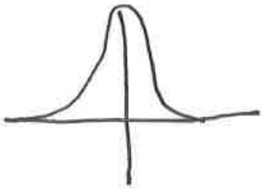
✓ double angle stuff.

Ex 24.9 : Fourier inversion says

$$\int_{-\infty}^{\infty} \left(\frac{\sin(\pi\lambda)}{\pi\lambda} \right)^2 e^{2\pi i \lambda x/a} d\lambda = \begin{cases} 1 - |x|/a & |x| \leq a. \\ 0 & |x| > a \end{cases}$$

This integral is probably impossible to calculate w/ normal calculus!

Ex 24.10 : (Fourier transform of Gaussian)



$f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-x^2/2\sigma^2}$ is

$$\hat{f}(\lambda) = \int_{-\infty}^{\infty} e^{-x^2/2\sigma^2} e^{-i\lambda x}\, dx \cdot \frac{1}{\sigma\sqrt{2\pi}}$$

$$= \int_{-\infty}^{\infty} e^{-\frac{1}{2\sigma^2}[x^2 + \overset{2\sigma^2}{i}\lambda x]} dx \cdot \frac{1}{\sigma\sqrt{2\pi}}$$

Complete the square

$$= \int_{-\infty}^{\infty} e^{-\frac{1}{2\sigma^2}[(x+i\frac{\lambda}{2})^2 + \frac{\lambda^2(2\sigma^2)^2}{4}]} dx \downarrow \quad (x + \frac{\lambda}{2} i)^2 + \frac{\lambda^2}{4} = x^2 + i\lambda x$$

$$= e^{-\lambda^2(2\sigma^2)^2/2\sigma^2\cdot 4} \underbrace{\int_{-\infty}^{\infty} e^{-\frac{1}{2\sigma^2}[(x+i\frac{\lambda}{2})]^2} dx}_{u = x + \frac{i\lambda}{2}} \cdot \frac{1}{\sigma\sqrt{2\pi}}$$

$$= \sigma\sqrt{2\pi}$$

$$= e^{-\lambda^2\sigma^2/2}$$

⇒ Fourier transform of gaussian is gaussian w/ $1/\sigma$ standard deviation

Rem 24.11 : This is a manifestation of Heisenberg's uncertainty principle.

If position/spatial profile is narrow then frequency/momentum must be broad! (Ex. on ultrafast laser pulses).

24.iii) Differentiation Under the Integral Sign

A fast shortcut to integration by parts not taught in calculus.

Ex 24.12 : Compute $\int_0^\infty x^4 e^{-x} dx$ ↙ integrate by parts 4 times? no thanks.

Set b=1.

$$= \int_0^\infty x^4 e^{-bx} dx$$

$$= \int_0^\infty \left(\frac{d}{db}\right)^4 e^{-bx} dx$$

$$= \left(\frac{d}{db}\right)^4 \int_0^\infty e^{-bx} dx = \left(\frac{d}{db}\right)^4 \left[-\frac{1}{b}e\right]_0^\infty = \left(\frac{d}{db}\right)^4 \left[-\frac{1}{b}\right] = \frac{24}{b^5}.$$

⇒ Makes many computations of Fourier transform easier.

Ex 24.13 : Display Fourier table.

Lecture 25 | Fourier Transforms II : Applications to ODEs + PDEs.

Recall

$$\hat{f}(\lambda) = \int_{-\infty}^{\infty} f(x) e^{-i\lambda x}\, dx.$$

is the Fourier transform of f.

25.i) Properties of the Fourier transform

The Fourier transform obeys

Prop 25.1 (Properties of Fourier transform 1)

i) $\widehat{f+g} = \hat{f} + \hat{g}$

ii) $\widehat{\alpha f} = \alpha \hat{f}$

iii) $\widehat{f(x+a)} = e^{i\lambda a} \widehat{f(x)}$,

iv) $\widehat{e^{ibx} f(x)} = \hat{f}(\lambda - b)$.

First two obvious,

$$\int f(x+a) e^{-i\lambda x} dx \underset{\overrightarrow{y=x+a}}{=} \int f(y) e^{-i\lambda(y-a)} dy = e^{i\lambda a} \int f(y) e^{-i\lambda y} dy$$

$$\int f(x) e^{ixb} e^{-i\lambda x} dx = \int f(x) e^{-i\lambda' x} dx \qquad \lambda = \lambda - b$$

v) $\widehat{f(cx)} = \frac{1}{c} \hat{f}\left(\frac{\lambda}{c}\right)$.

Ex 25.2: Let $f = \begin{cases} 1 & |x| \leq 1 \\ 0 & \text{else}. \end{cases}$ Last time we showed $\hat{f} = \frac{2\sin\lambda}{\lambda}$

Compute $\hat{g}$ for $g = \begin{cases} 2 & 3 \leq x \leq 11 \\ 0 & \text{else}. \end{cases}$

$$g = 2f\left(\frac{x-7}{4}\right) = 2\, \widehat{f\left(\tfrac{x-7}{4}\right)} = 2 \cdot \left(\tfrac{1}{4}\right)^{-1} \widehat{f(x-7)}\left(\lambda \cdot 4\right)$$

$$= 2 \cdot 4 e^{-i7\lambda} \cdot \hat{f}(4\lambda)$$

$$= 8 e^{-i7\lambda} \cdot \frac{2\sin(4\lambda)}{4\lambda} = 4 e^{-7i\lambda} \frac{\sin(4\lambda)}{\lambda}.$$

25.ii) Fourier Transform and Differentiation

Key idea 25.3: Fourier transform turns differentiation into multiplication!

Thm 25.4: (Fourier differentiation)

- $$\widehat{\frac{d}{dx}f} = i\lambda \hat{f}(\lambda)$$
- $$\widehat{-ixf} = \frac{d}{d\lambda}\hat{f}(\lambda)$$

Because

$$\widehat{\frac{d}{dx}f} = \int_{-\infty}^{\infty} \frac{d}{dx}f\, e^{-i\lambda x}\,dx = -\int_{-\infty}^{\infty} f \frac{d}{dx} e^{-i\lambda x} dx + \cancel{f e^{-i\lambda x}\Big|_{x=-\infty}^{x=\infty}}$$

$= 0$ b/c $\int|f| < \infty$.

$$= i\lambda \int_{-\infty}^{\infty} f(x)\, e^{-i\lambda x} dx$$
$$= i\lambda \hat{f}.$$

Ex 25.5: (Fourier transform reproduces characteristic polynomial)

Solve
$$\frac{d^2f}{dx^2} + k^2 f = 0$$
using the Fourier transform.

If

$$\frac{d^2f}{dx^2} + k^2 f = 0 \Rightarrow \mathcal{F}\left(\frac{d^2f}{dx^2} + k^2 f\right) = 0 = \mathcal{F}(0)$$
$$\mathcal{F}\left(\frac{d^2f}{dx^2}\right) + k^2\mathcal{F}(f) = 0.$$
$$-\lambda^2 \hat{f} + k^2 f(\lambda) = 0 \quad \text{for all } \underline{\lambda \in \mathbb{R}}$$

$(\lambda^2 - k^2)\hat{f}(\lambda) = 0 \Rightarrow$ either $\hat{f}(\lambda) = 0$ or
$\lambda^2 - k^2 = 0.$

$\Rightarrow \hat{f} =$ 

$\Rightarrow f = C_1 e^{ikx} + C_2 e^{-ikx}.$

Ex 25.6: Solve $\left(\frac{d}{dx}\right)^n f + a_{n-1}\left(\frac{d}{dx}\right)^{n-1} f + \dots + a_0 f = 0.$

Take $\mathcal{F}$ of both sides

$$\left[(i\lambda)^n + a_{n-1}(i\lambda) + \dots + a_0\right]\hat{f}(\lambda) = 0 \Rightarrow$$ Characteristic polynomial for $(i\lambda)$.

Ex 25.7 (Inhomogeneous Equation)

Solve $$a\left(\frac{d}{dx}\right)^2 f + b\frac{d}{dx}f + cf = e^{ix\cdot\gamma}$$

$$a(i\lambda)^2\hat{f} + b(i\lambda)\hat{f} + c\hat{f} = \hat{g} \quad \rightarrow \quad \delta(\lambda-\gamma) = \begin{cases} \infty & \lambda=\gamma \\ 0 & \text{else} \end{cases}$$

$$\hat{f} = \frac{\hat{g}(\lambda)}{a(i\lambda)^2 + b(i\lambda) + c}$$

$$\hat{f} = \delta \frac{1}{a(i\gamma)^2 + b(i\gamma) + c}\hat{g}(\lambda)$$

Apply $\mathcal{F}^{-1}$ →

$$f = \frac{e^{ix\cdot\gamma}}{a(i\gamma)^2 + b(i\gamma) + c}$$

(check: obviously $\left(a\frac{d^2}{dx^2} + b\frac{d}{dx} + c\right)\frac{e^{ix\gamma}}{\cdots} = e^{ix\gamma}$.)

General Philosophy 25.8

$$\begin{tikzcd}
\left\{ \begin{array}{l} Lf = g \\ \text{linear ODE or PDE} \end{array} \right\} \arrow[r, "\mathcal{F}"] \arrow[d, "\text{normal, direct solving methods}"'] \arrow[dr, phantom, "\text{Fourier solving approach}"] & \left\{ \begin{array}{l} P(\lambda)\hat{f} = \hat{g} \\ \text{linear poly. equation} \end{array} \right\} \arrow[d, "\text{easy to solve } \hat{f} = \frac{\hat{g}(\lambda)}{P(\lambda)}"] \\
\{ \text{solution } f \} & \left\{ \hat{f} = \frac{\hat{g}(\lambda)}{P(\lambda)} \right\} \arrow[l, "\mathcal{F}^{-1}"']
\end{tikzcd}$$

Ex 25.9: Solve

$$a\left(\frac{d}{dx}\right)^2 f + b\frac{d}{dx}f + cf = g$$

Thm 25.10: The general solution to the inhomogeneous equation is

$$f(x) = \mathcal{F}^{-1}\left(\frac{\hat{g}(\lambda)}{a(i\lambda)^2 + b(i\lambda) + c}\right) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{\hat{g}(\lambda)}{a(i\lambda)^2 + b(i\lambda) + c} e^{i\lambda x}\, d\lambda$$

Provided the denominator has no real routes, whether we can compute that integral is a separate question, but it is now in theory just a math 20 problem.

25. iii) Fourier transform and the heat equation

The same idea applies to solving PDEs like the heat equation except the problem reduces

$$\text{PDE} \rightsquigarrow \text{ODE}$$

instead of

$$\text{ODE} \rightsquigarrow \text{Algebraic eq.}$$

Def 25.11: The IVP for the heat equation is (on $\mathbb{R} \times [0,\infty)$).

$$\begin{cases} \frac{\partial U}{\partial t} = \alpha \nabla^2 U \\ U(0,x) = g(x) \end{cases}.$$

Note there is no boundary condition at $x = \pm\infty$ (this becomes the requirement that $\int |u| < \infty$ so $U \to 0$ at ends).

Taking transforms,

$$\widehat{\frac{\partial}{\partial t} U} = \alpha \widehat{\left(\frac{\partial}{\partial x}\right)^2 U}$$

Note $\widehat{\partial_t U} = \partial_t \hat{U}$ (∂_t just pulls through the Fourier integral).

$$\frac{d}{dt} \hat{U}(\lambda,t) = -\alpha\lambda^2 \hat{U}(t,\lambda).$$

(an ODE for each λ).

$$\hat{U}(\lambda,t) = \hat{U}(0,\lambda) \cdot e^{-\alpha\lambda^2 t}$$

$$U(t,x) = \mathcal{F}^{-1}\left(\hat{U}(0,\lambda) e^{-\alpha\lambda^2 t}\right)$$

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} \hat{g}(\lambda) e^{-\alpha\lambda^2 t} e^{i\lambda x} \, d\lambda.$$

Rem 25.12: There is an analogy with decoupling again. For different x, $\nabla^2 U$ depends on x in a small interval or just at x. But

$-\alpha\lambda^2 \hat{U}(t,\lambda)$ is "diagonal"

ie different values of λ don't talk to eachother. This analogy can be made precise by noting that $\mu = \lambda^2$ $v = e^{i\lambda x}$ and an eigenvector/value

$$\nabla^2 v = \mu v$$

pair for the Laplacian.

Lecture 26 Convolution + Integral Formula Solutions

Recall that last time we showed the general solution of the heat equation

$$\begin{cases} \frac{\partial U}{\partial t} = \alpha \nabla^2 U \\ U(0,x) = g(x) \end{cases}$$

Was $$U(t,x) = \mathcal{F}^{-1}\left(\hat{g}(\lambda) e^{-\alpha\lambda^2 t}\right).$$

Question 26.1: What is the Fourier transform (or inverse) of a product.

Def 26.2: For two functions $f,g : \mathbb{R} \to \mathbb{C}$ define the convolution by

$$(f*g)(x) = \int_{-\infty}^{\infty} f(x-y)g(y)\,dy$$

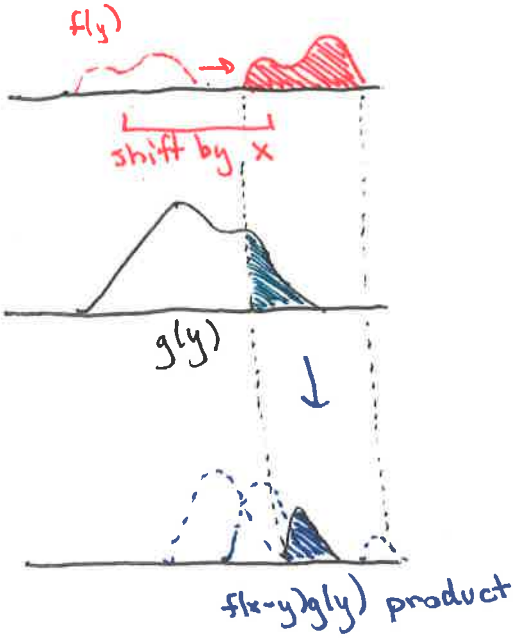
Note y is integrated away, so the remainder is a function of x.

Thm 26.3 (Fourier Product)

$$\widehat{f(x)g(x)} = (\hat{f} * \hat{g})(\lambda)$$

$$\widehat{(f*g)(x)} = \hat{f}(\lambda)\hat{g}(\lambda).$$

26.i) Intuition for the convolution



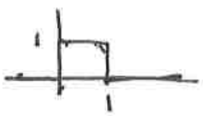
As red $f(x-y)$ is translated, the "overlap" profile in purple changes.

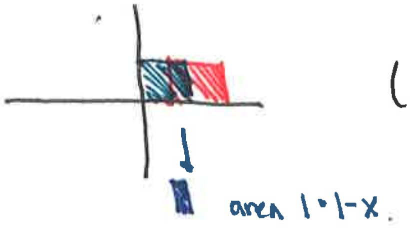
$\Rightarrow$ For each x,

$(f*g)(x) = \Big\{$ Area under  product for translate by x $\Big\}$.

Ex 26.3: Find the convolution $(f*f)(x)$

when $f(x) =$



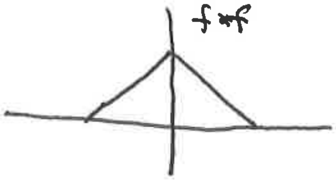


$$(f*f)(x) = \int_{-\infty}^{\infty} f(x-y)\,f(y)\,dy$$

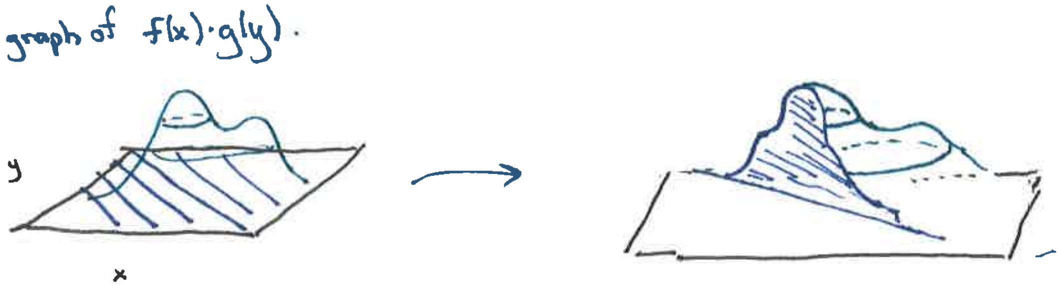
$$= \int_0^1 f(x-y)\,dy$$

$$= \int_{\cancel{0}\,x}^{1} 1\,dy = 1-x \qquad \text{(same for negative}$$

$$= \begin{cases} 1-|x| & |x| \leq 1 \\ 0 & \text{else.} \end{cases}$$

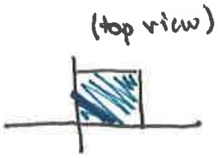


Rem 26.4: Another way to visualize

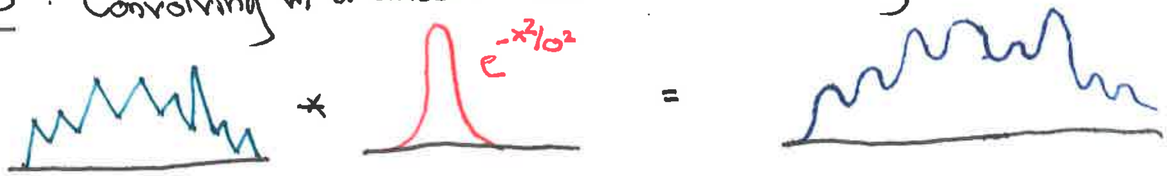


The convolution at x is the integral of the diagonal slice along line $\{(x-y,y) \mid y \in \mathbb{R}\}$ (purple).

In Ex 26.3:



Rem 26.5: Convolving w/ a smooth function has a smoothing effect



- See "Gaussian blur" tool in photo editing/drawing
- "Convolutional Neural Networks" allow layers of neurons to communicate by smearing out inputs from previous layer(s) via a convolution.

The following operations are useful for computing with convolutions

Prop 26.6

- $$(f*g)(x) = \int_{-\infty}^{\infty} f(x-y)g(y)dy \underset{z = x-y}{=} -\int_{-\infty}^{\infty} f(z)g(x-z)$$ (change of variables)

- $$\frac{\partial}{\partial x} f(x-y) = -\frac{\partial}{\partial y} f(x-y) \qquad \int_{-\infty}^{\infty} \underbrace{\frac{\partial}{\partial x}}_{\text{pulls out of integral}} f(x-y)g(y)dy = -\int_{-\infty}^{\infty} \underbrace{\frac{\partial}{\partial y}}_{\text{Integrates by parts.}} f(x-y)g(y)dy$$

26.ii) Convolution and the Heat Kernel

Recall that the solution to the IVP $\begin{cases} \frac{\partial U}{\partial t} = \alpha \nabla^2 U \\ U(0,x) = g(x) \end{cases}$ is

$$U(t,x) = \mathcal{F}^{-1}\left(\hat{g}(\lambda) \cdot e^{-\alpha\lambda^2 t}\right).$$

$$= \mathcal{F}^{-1}(\hat{g}(\lambda)) * \mathcal{F}^{-1}(e^{-\lambda^2 \alpha t}).$$

Recall $\mathcal{F}\left(\frac{1}{\sqrt{2\pi}\sigma} e^{-x^2/2\sigma^2}\right) = e^{-\lambda^2\sigma^2/2}$ so $\sigma^2/2 = \alpha t$

$$\mathcal{F}\left(\frac{1}{\sqrt{4\pi\alpha t}} e^{-x^2/4\alpha t}\right) = e^{-\lambda^2 \alpha t}.$$

Obviously $\mathcal{F}^{-1}(\hat{g}(\lambda)) = \mathcal{F}^{-1}\mathcal{F}(g) = g$. Combining this

Thm 26.7 (Heat Kernel Solution) The (unique) solution of the IVP

$$\begin{cases} \frac{\partial U}{\partial t} = \alpha \nabla^2 U \\ U(0,x) = g(x) \end{cases}$$

is given by

$$U(t,x) = \int_{-\infty}^{\infty} g(y) \cdot \frac{1}{\sqrt{4\pi\alpha t}} e^{-(x-y)^2/4\alpha t} dy$$

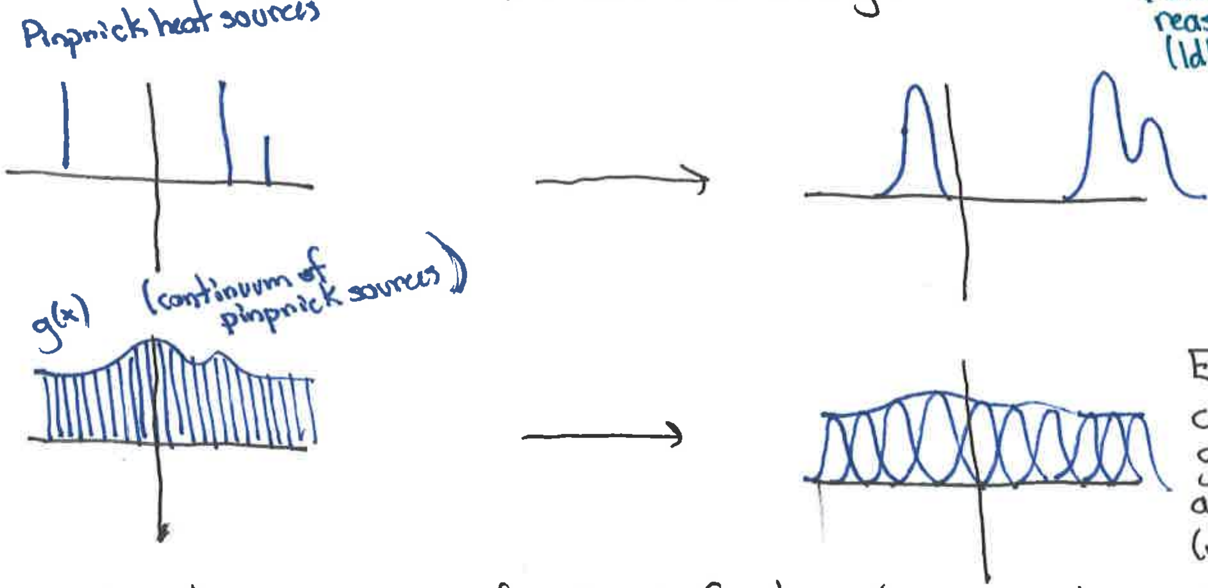
$$= (g * K_t)(x).$$

where $K_t(x) = \frac{1}{\sqrt{4\pi\alpha t}} e^{-x^2/4\alpha t}$ is the Heat Kernel.

The operation of convolving with K_t is called the integral Kernel method.

Rem 26.8 : Kernel (noun): 1) the central or essential part of something
2) the seed of a cereal grain

One of these defs is the reason for the terminology (Idk which)



Each point in the continuum source g(x) spreads as a gaussian (with no mutual interaction)

The heat kernel is an example of a Green's function (and is sometimes called that)

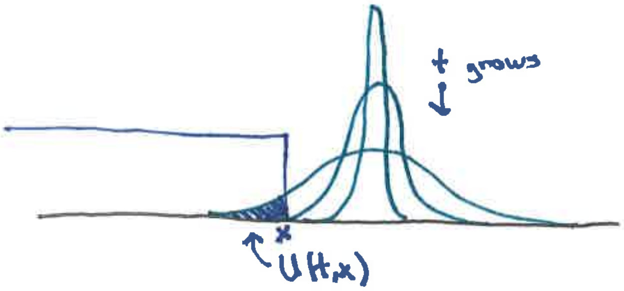
double integral guy

Ex 26.9 : Let $g(x) = \begin{cases} 1 & x \geq 0 \\ 0 & \text{else} \end{cases}$. Solve the IVP for the heat equation.

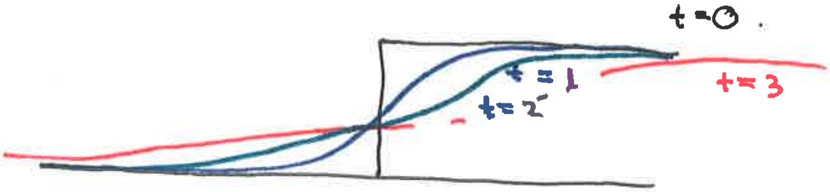
$$U(t,x) = \frac{1}{\sqrt{4\pi\alpha t}} \int_{-\infty}^{\infty} g(y)\, e^{-(x-y)^2/4\alpha t}\, dy = \frac{1}{\sqrt{4\pi\alpha t}} \int_{-x}^{\infty} e^{-z^2/4\alpha t}\, dz$$

$z = y - x$.

$$= \frac{1}{\sqrt{4\pi\alpha t}} \int_{-\infty}^{x} e^{-w^2/4\alpha t}\, dw.$$



"error function" ie int of bellcurve.



curves are $\mathrm{Erf}_{\sqrt{4\alpha t}}(x) = \int_{-\infty}^{x} e^{-w^2/4\alpha t}\, du$

Thm 26.10 : Quantum Mechanics is wrong.

Proof : Observe for any $t>0$, heat is found arbitrarily far left.
Same for Schrödinger equation ($i\frac{\partial U}{\partial t} = \alpha\nabla^2 U$), but this violates relativity (c = speed limit)
$\Rightarrow$ motivated development of QFT (Dirac).

26.iii) Comparison of Fourier + Integral Kernel methods

Recall the "decoupling" method was to find a diagonal basis, solve in basis, then add up solutions (superposition)

Two bases of $\mathbb{C}$-valued functions

"e_i"

1) Frequency Basis $e^{i\lambda x}$, $\lambda \in \mathbb{R}$

2) "Position" basis $K_0(x-a)$ $y \in \mathbb{R}$

$$K_0(x-a) = \begin{cases} \infty & x=a \\ 0 & \text{else} \end{cases}$$

Write in basis $f = \sum \langle f, e_i \rangle e_i$

$$\sum_{\lambda} \hat{g}(\lambda) e^{i\lambda x}$$

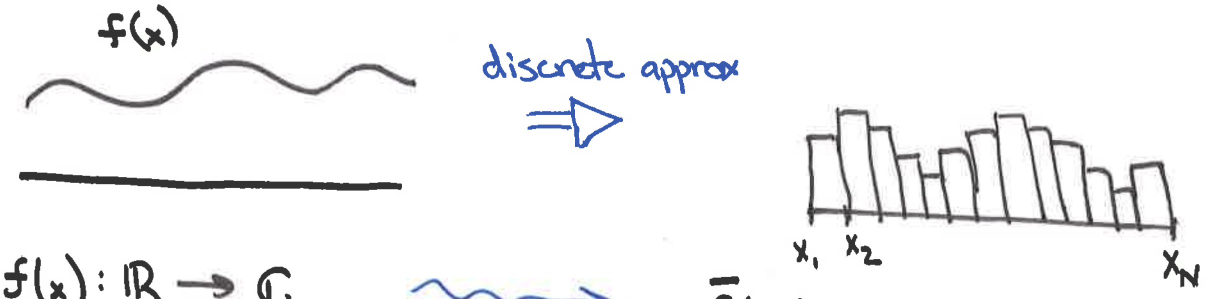
$$\sum_{y} g(y) K_0(x-y)$$

since basis is continuum not discrete $\sum$ becomes $\int \; d\lambda$.

Lecture 27 The Fast Fourier transform + Spectral Methods

Just like for ODEs, most difficult PDEs must be solved numerically sometimes using the Fourier transform.

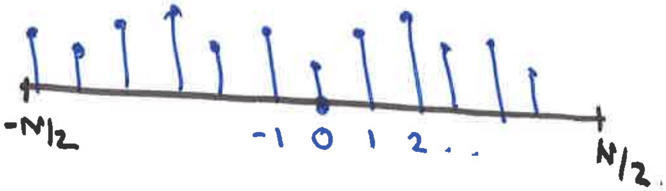
27.i) The Discrete and Fast Fourier transforms



$f(x): \mathbb{R} \to \mathbb{C} \quad \rightsquigarrow \quad \bar{f}(x_i): \mathbb{R}^N \to \mathbb{C}.$

Similarly on Fourier side

$$f(x) = \sum_{n=-\infty}^{\infty} \hat{f}(n) e^{inwx} \quad \rightsquigarrow \quad \cancel{\hat{f}}\ \bar{f} \approx \sum_{n=N/2}^{N/2} \hat{f}(n) e^{inwx}$$



For all N fourier coefficients,

$$\hat{f}(n) = \int_{-L}^{L} f(x) e^{-inwx} dx$$

$$\approx \sum_{k=1}^{N} \bar{f}(x_k) e^{-inwx_k} \quad \text{(Riemann sum)}$$

$= \mathcal{O}(N)$ operations,

$f(x) \approx \sum_{n=-N/2}^{N/2} \hat{f}(n) e^{-inwx}$ requires $\mathcal{O}(N \cdot N) = \mathcal{O}(N^2)$ operations.

Ex 27.1: WiFi uses Discrete Fourier transforms to convert sensor/reciever input to data, in blocks of 2048 bits.

~ 4 million computations for single block.

$\boxed{\text{Thm } 27.2}$ FFT

Given $\bar{f}(x_i)$ a discrete function with resolution N, the discrete Fourier transfor
$\hat{f}(n)$ for $|n| \leq \frac{N}{2}$ can be computed by an algorithm requiring only

$$\sim N \cdot \log N$$

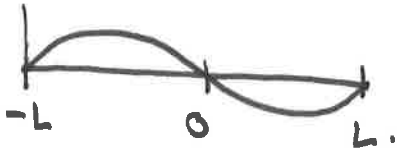
operations. It is called the Fast Fourier Transform.

Math in the World 27.3 : top 10 algorithms.

Idea 27.4 : Write

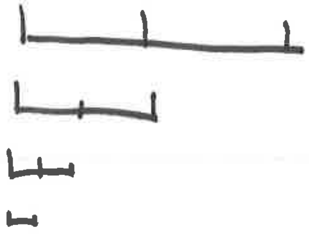
$$f(x) = \sum_{\text{even}} \hat{f}(2m) e^{i\,2mx} + \sum_{\text{odd}} \hat{f}(2m+1)\, e^{i(2m+1)x}$$

biggest period



$\bar{f}(x + \frac{N}{2}\Delta x)$ can be obtained from $\bar{f}(x)$ for x in $[-L, 0]$

$\Rightarrow$



recursive subdivision.

Ex 27.5 : Wifi blocks of 2048 now require

$$\sim 2048 \cdot \log 2048 \approx 10{,}000$$

operations. $\Rightarrow$ Wifi could not be more than Kb/s (now Gb/s) w/o FFT!!

Secret 27.6

27.ii) Spectral Methods

Consider a PDE

$$\frac{\partial u}{\partial t} = a_n \frac{\partial^n u}{\partial x^2} + \cdots + a_0 u$$

A **spectral method** is a numerical method using Fourier methods.

Approximate $u = \sum_{-N}^{N} \hat{u}(t,n) e^{inx}$ (calculated w/ FFT)

$$\frac{d}{dt}\begin{pmatrix} \hat{u}_{-N} \\ \vdots \\ \hat{u}_N \end{pmatrix} = \begin{pmatrix} & & \\ & \hat{A} & \\ & & \end{pmatrix} \begin{pmatrix} \hat{u}_{-N} \\ \vdots \\ \hat{u}_N \end{pmatrix}$$

Ex 27.7 (Heat Eq) In the Fourier basis, $\frac{\partial^2 u}{\partial x^2}(e^{ikx}) = -k^2$

$\frac{\partial u}{\partial t} = \nabla^2 u$ becomes

$$\frac{d}{dt}\begin{pmatrix} \hat{u}_{-N} \\ \vdots \\ \hat{u}_N \end{pmatrix} = \begin{pmatrix} -4 & & & & & & \\ & -1 & & & & & \\ & & 0 & & & & \\ & & & -1 & & & \\ & & & & -4 & & \\ & & & & & -9 & \\ & & & & & & \ddots & \\ & & & & & & & -N^2 \end{pmatrix} \overset{\text{very stiff!}}{\begin{pmatrix} \hat{u}_{-N} \\ \vdots \\ \hat{u}_N \end{pmatrix}}.$$

Ex 27.8: $\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + 2\cos(x) u$

$$\frac{d}{dt}\begin{pmatrix} \hat{u}_{-N} \\ u_{-N} \end{pmatrix} = Au + Bu$$

B = discretized $2\cos * u$

$$= \begin{pmatrix} 0 & 1 & & & \\ 1 & 0 & 1 & & \\ & 1 & 0 & 1 & \\ & & 1 & \ddots & \\ & & & & \ddots \end{pmatrix}$$

Thm 27.9: The different

$$\|u(t,x) - \bar{u}(t,x)\| \leq Ce^{-N}$$

approaches the true sol. *exponentially* in # of Fourier modes!

Rem 27.10: This is a numerical cheatcode, but only one domains w/ nice geometry.

27. iii) What next?

Math 53
ODEs

Physics
- Quantum Mechanics (Phy 71, Matsci 201)
- Electromagnetism (Phy 128 ..).

Math
- Math 131 (PDEs)
- Math 158 (Stochastic ODEs)
- Math 115/170s (Functions + Analy

ME/EE
- Dynamical Systems + Control (ME 161)
- Signals + Systems (EE 102)
- Aircraft design/orbital mechanics (AA 146/179)
- Fluid Dynamics (ME 70, AA 210, ...)

CS
- Scientific computing (CME 108)
- Machine Learning in Physical Sciences (CME 215)

Numerical + Computation
- Computer lin. alg (Math 104, CME 200)
- Numerical Methods (Math 220 A, B, CME 104, 204).
- Dynamical Systems (CME 263, /EE)