Lecture 1 Motivation for measure theory

1.i) Syllabus + Introduction

1.ii) Motivation for Real Analysis

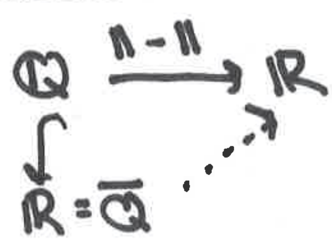
Real analysis is the rigorous foundation of calculus, convergence, and functions.

Applications 1.1: Applications of real analysis include

1) Description of physical systems by ODEs and dynamical systems
2) Study of partial differential equations (PDEs) describing Electromagnetism, heat, waves, quantum particles, quantum fields, fluids, plasmas, ..
3) Analytic Number theory + Automorphic forms
4) Representation theory and harmonic analysis
5) Probability theory
6) Optimization problems in physics, math, economics, etc.
7) Study of fractals and chaotic phenomena
8) Dynamics and ergodic theory

(Informal) Def 1.2: A space X is said to be complete if limits of $x_i \in X$ converge to points of X.

Ex 1.3: The rational numbers $\mathbb{Q}$ are an incomplete (metric) space, i.e. not all Cauchy sequences converge. To fix this is the main motivation for introducing real numbers.



recall distance of x to $y \in \mathbb{Q}$ is $\|x-y\| = |x-y|$.

metric completion

(yes this example is circular, its just motivation)

➡ in $\mathbb{R}$ every Cauchy sequence converges.

<u>Motivation 1.9</u> (Fourier Series)

Fourier Series gives a way of expressing a (periodic) function as a sum

$$f(x) = \sum_{n=0}^{\infty} a_n \cos(nx) + b_n \sin(nx).$$

What does it mean for this to converge? $f_N = \sum^{N}$ then

$f_N(x) \to f(x)$ for every x (pointwise convergence)

$\sup |f_N(x) - f(x)| \to 0$ (uniform convergence)

$\int |f_N(x) - f(x)| \to 0$ (Integral convergence).

When are each true? For what sequences $\{a_n, b_n\}$ is $f(x)$ continuous? Differentiable.

Studying these questions in the context of PDEs was historically a main motivation for measure theory.

Motivation 1.4: We can define a metric on the space of continuous functions $C^0[a,b]$ on an interval $[a,b] \subseteq \mathbb{R}$ by

$$\|f-g\| = d(f,g) = \int_a^b |f-g|\,dx \quad \text{(Riemann integral)}$$

But this space is certainly not complete!

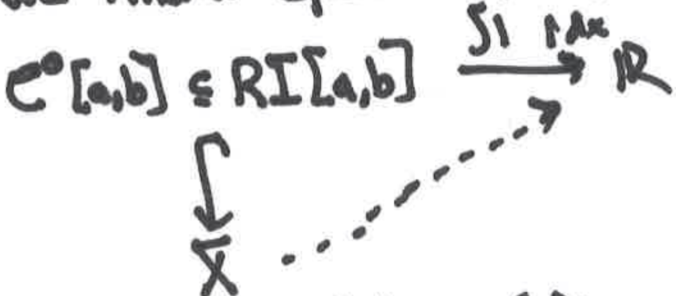
Ex 1.5:

Difference of integrals $\leq \frac{1}{2}bh \to 0$.

Ex 1.6: Consider $F(x) = \begin{cases} 1 & x \in \mathbb{Q} \\ 0 & x \in \mathbb{R} \setminus \mathbb{Q}. \end{cases}$

Then $F(x)$ is not Riemann integrable! But it is a limit of these (more next time).

Question 1.7: Can we find a space $\bar{X}$ such that



is the metric completion of Riemann integrable functions?

A: Yes! It is $L^1[a,b]$ the Lebesgue integrable functions.

Motivation 1.8: If f_n is a sequence of functions, when does

$$\lim_{n\to\infty} \int_a^b f_n(x)\,dx = \int_a^b \lim_{n\to\infty} f_n(x)\,dx \;?$$

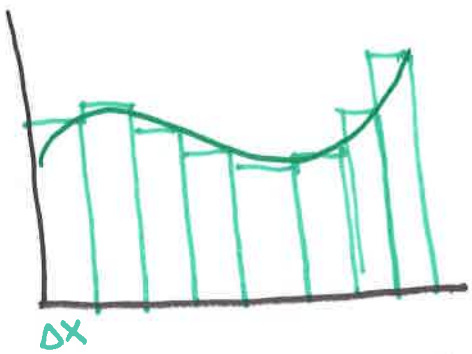
This is very annoying w/ Riemann (and RHS may not exist!).

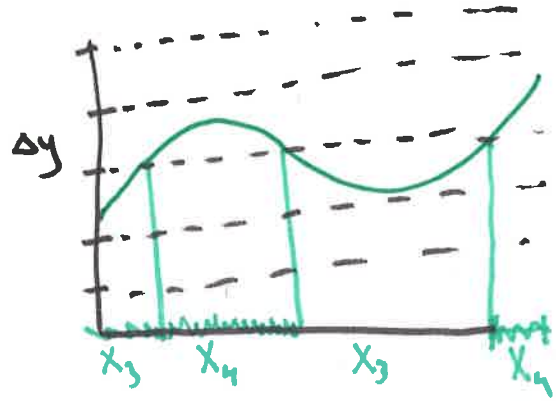
It becomes easy with Lebesgue Dominated Convergence Thm (Lecture 9).

Lecture 2 σ-algebras and measures

Ex 2.1 : Coin counting example.

2.i) Intuition for the Lebesgue Integral





Divide up x-axis, then

$$\int_0^1 f\,dx = \lim_{\Delta x \to 0} \sum_{i=1}^{N} \Delta x_i \cdot f(x_i)$$

$$\int_0^1 f\,dx = \lim_{\Delta y \to 0} \sum_{j=1}^{N} j\Delta y \cdot m(X_j)$$

"size or measure of X_j"

Goal 2.2 : Define a measure

$$m : \{A \mid A \subseteq \mathbb{R}^n \text{ is a subset}\} \to \mathbb{R} \cup \{\infty\}.$$

that satisfies "reasonable" properties.

1) $m([a,b]) = b - a$ (normalization)
2) $m(A \cup B) = m(A) + m(B) - m(A \cap B)$ (well-behaved under $\cup, \cap$)
3) $m(A + x) = m\{a + x \mid a \in A\} = m(A)$ (translation invariance)

Rem 2.3 : $\nexists\, m$ satisfying the above for all subsets

$$\left\{\begin{matrix}\text{finite unions}\\ \text{of intervals}\end{matrix}\right\} \subsetneq \left\{\begin{matrix}\text{Borel}\\ \text{sets}\end{matrix}\right\} \subsetneq \left\{\begin{matrix}\text{measurable}\\ \text{subsets}\end{matrix}\right\} \subsetneq \left\{\text{All subsets}\right\}$$

$$\text{measurable subsets} \xrightarrow{\ m\ } \mathbb{R} \cup \{\infty\}.$$

Rem 2.12 : Every $x \in \mathbb{R}$ has a continued fraction expression

$$x = a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{\ddots}}}$$

w/ $a_0 \in \mathbb{Z}$ and $a_i \in \mathbb{N}\ \mathbb{Z}^{\geq 0}$.

$K = \{x \in \mathbb{R} \mid x \notin \mathbb{Q}$ and $(a_0, a_1, \dots)$ has a subsequence a_{i_k} st $a_{i_k} \mid a_{i_{k+1}}\}$.

is not a Borel set.

2.iii) Measures :

Def 2.13 : Let (X, Σ) be a metric space with σ-algebra. A measure

$$\mu : \Sigma \longrightarrow \mathbb{R} \cup \{\infty\}$$

is a function such that

i) $\mu(E) \geq 0 \ \forall E \in \Sigma$

ii) $\mu(\emptyset) = 0$

iii) If $\{E_i\}_{i=1}^{\infty}$ are a countable disjoint collection

$$\mu\left(\bigcup_{i=1}^{\infty} E_i\right) = \sum_{i=1}^{\infty} \mu(E_i).$$

Def 2.14 : A measure space is a triple (X, Σ, μ) of a metric space and σ-algebra with a measure.

It is complete if $E \subseteq A \in \Sigma$ then $\mu(A) = 0 \Rightarrow \mu(E) = 0$ ($E \in \Sigma$).

Theorem 2.15 : There exists a measure m and σ-algebra $\mathcal{M}(\mathbb{R}^n)$ making $(\mathbb{R}^n, \mathcal{M}(\mathbb{R}^n), m)$ into a measure space with the addition properties

0) $m([a,b]) = b - a$

i) m is complete

ii) If $A \in \mathcal{M}(\mathbb{R}^n)$, $A + x \in \mathcal{M}(\mathbb{R}^n)$ and $m(A) = m(A + x)$.

iii) $\mathcal{B}(\mathbb{R}^n) \subseteq \mathcal{M}(\mathbb{R}^n)$.

m is called the Lebesgue measure, and $\mathcal{M}(\mathbb{R}^n)$ the measurable sets.

Note I_x, I_y distinct or equal. If ~~not~~ $x \neq y$, then $I_x \cup I_y$ ~~open~~, is an interval contradicting maximality unless $I_x = I_y$. And each I_x is open so contains some $q_i \in \mathbb{Q}$. Since the collection of distinct intervals contains distinct rationals, ~~they~~ it is countable □

Lemma 2.8: Every open $U \subseteq \mathbb{R}^d$ is a union

$$U = \bigcup_{n=1}^{\infty} [a_1^n, b_1^n] \times \dots \times [a_d^n, b_d^n]$$

that have disjoint interiors.

Proof: Exercise (see pages 7-8).

Let $P(X)$ denote the power set $P(X) = \{A \mid A \subseteq X\}$.

Definition 2.9. A σ-algebra on X is a subset

$$\Sigma \subseteq P(X)$$

such that

i) $X \in \Sigma$

ii) $A \in \Sigma \Rightarrow A^c \in \Sigma$ (complements)

iii) $A_i \in \Sigma$ $i = 1, \dots \Rightarrow \bigcup_{i=1}^{\infty} A_i \in \Sigma$ (countable unions).

(same for intersections by ii $\bigcap A_i = (\bigcup A_i^c)^c$.

Def 2.10: If X is a metric space, the σ-algebra

$$\mathcal{B}(X) = \{\text{small } \sigma\text{-algebra containing open sets}\} = \bigcap_{\text{Opens} \subseteq \Sigma} \Sigma \quad \leftarrow P(X) \text{ is a } \sigma\text{-algebra so non-empty.}$$

is called the Borel sets of X.

Rem 2.11:

$$\left\{\begin{matrix}\text{open/closed} \\ \text{sets}\end{matrix}\right\} \subsetneq \left\{\begin{matrix}\text{Borel} \\ \text{subsets}\end{matrix}\right\} \subsetneq P(\mathbb{R})$$

- $\left\{\text{Borel subsets}\right\}$ ← has cardinality of $\mathbb{R}$
- $P(\mathbb{R})$ ← bigger cardinality "most sets are not Borel".
- $[0, 1/2) = \bigcup$ closed (showing open/closed sets $\subsetneq$ Borel subsets)

Ex 2.4 : Consider $f : [0,1] \longrightarrow \mathbb{R}$

$$f(x) = \begin{cases} 1 & x \in \mathbb{Q} \cap [0,1] \\ 0 & \text{else} \end{cases}$$

Claim : $f(x)$ is not Riemann intigrable

$$\left| \sum_{i=1}^{N} \inf_{[x_i, x_{i+1}]} f(x) \cdot \Delta x - \sum \sup_{[x_i, x_{i+1}]} f(x) \cdot \Delta x \right| = 1$$

For all $\Delta x > 0$.

But in a Lebesgue sense

$$\int_0^1 f \, dx = 1 \cdot \underbrace{m(\mathbb{Q} \subseteq [0,1])}_{=0.}$$

$\forall \varepsilon > 0$, $m(\mathbb{Q} \cap [0,1]) \leq \varepsilon$. Let $\{q_n\}_{n \in \mathbb{N}}$ be an enumeration of $\mathbb{Q}$.

Let $I_n = \left[q_n - \frac{\varepsilon}{2^{n+1}}, q + \frac{\varepsilon}{2^{n+1}}\right]$. Then $\mathbb{Q} \cap [0,1] \subseteq \bigcup I_n$

but $m(\bigcup I_n) \leq \sum_{n=1}^{\infty} m(I_n) - m(\cap \ldots) \leq \sum_{n=1}^{\infty} m(I_n) = \varepsilon \sum_{n=1}^{\infty} \frac{1}{2^n}$

$= \varepsilon$ ∎.

2.ii) Open and Borel sets, σ-algebras.

Def 2.5 : An open subset $U \subseteq \mathbb{R}$ is one such that $\forall x \in U, \exists r_x$ st $B_{r_x}(x) \subseteq U$.

A closed subset C is one so that $C^c = \{x \in \mathbb{R} \mid x \notin C\}$ is open.

Facts 2.6 : 1) Arbitrary unions of open sets are open (or $\cap$ of closed are closed)

2) finite intersection of open sets is open ($\cup$ of closed sets is closed.

Lemma 2.7 : Every open $U \subseteq \mathbb{R}$ may be written

$$U = \bigsqcup_{n=1}^{\infty} (a_n, b_n)$$

where $a_n, b_n = \pm\infty$ is allowed.

Proof : $\forall x \in U$, let $I_x = \left(\inf_{\substack{a \in U \\ \text{st } (a,x) \subseteq U}} a \, , \sup_{\substack{b \in U \\ \text{st } (x,b) \subseteq U}} b \right)$

$I_x \subseteq U$ is open, and $x \in I_x$, so $U = \bigcup I_x$.

Lecture 3 Outer Measure + Lebesgue Measure

Recall: We wish to construct a measure ("Lebesgue measure") and a σ-algebra $\mathcal{M}(\mathbb{R}^n)$ such that

i) $m(Q) = |Q|$ if $Q = [a_1, b_1] \times \ldots \times [a_n, b_n]$ is a cube

ii) m is complete and translation invariant

iii) $\mathcal{B}(\mathbb{R}^n) \subseteq \mathcal{M}(\mathbb{R}^n)$

Ex 3.1: It is easy to construct an arbitrary measure on $\Sigma = \mathcal{B}(\mathbb{R})$

Set
$$\mu(E) = \begin{cases} 1 & \text{if } 0 \in E \\ 0 & \text{else.} \end{cases}$$

Check $\mu(E)$ satisfies the properties of a measure.
This is called an "atomic" or "Dirac" measure.

Plan 3.2: First define "outer measure" on $P(\mathbb{R}^n)$

outer measure on $P(\mathbb{R}^n)$ ⇝ (= when defined.) Lebesgue measure on $\mathcal{M}(\mathbb{R}^n)$

Def 3.3: Let $E \subseteq \mathbb{R}^n$ be a subset. Let

$$m^*(E) = \inf \left\{ \sum_{j=1}^{\infty} |Q_j| \;\middle|\; E \subseteq \bigcup Q_j \text{ is a covering by cubes} \right\}.$$

Ex 3.4: $m^*(pt) = 0$ since $pt = x_0$, $Q = \prod [x_0^i - \frac{\varepsilon}{2}, x_0^i + \frac{\varepsilon}{2}]$ has volume $\varepsilon > 0$ for arbitrary ε.

Ex 3.5: $m^*(Q) = |Q|$ for a cube Q. Obvious Q covers itself so $m^*(Q) \le |Q|$. Must show $|Q| \le \sum_{j=1}^{\infty} |Q_j|$ for $Q \subseteq \bigcup Q_j$.

For each j, let $Q_j \subseteq S_j$ open w/ $|S_j| \le (1+\varepsilon)Q_j$ for $\varepsilon > 0$.

$$Q \subseteq \bigcup_{j=1}^{N} S_j \quad \text{by compactness,}$$

$$|Q| \le \sum_{j=1}^{N} |S_j| \le (1+\varepsilon) \sum_{j=1}^{N} |Q_j| \le (1+\varepsilon) \sum_{n=1}^{\infty} |Q_j|, \text{ take } \varepsilon \to 0$$ ∎

Ex 3.6 : $\mathring{Q}$ an open cube, $|\mathring{Q}| = m^*(Q)$ still holds.

Q covers, so $m^*(\mathring{Q}) \leq |Q|$. There is a closed $Q_0 \subseteq Q$ with volume $|Q_0| \geq (1-\varepsilon)|Q| \ \forall \varepsilon$, and

$$(1-\varepsilon)|Q| \leq |Q_0| \leq m(Q)$$

since any covering of Q covers Q_0. Now take $\varepsilon \to 0$.

Ex 3.7 : $m^*(\mathbb{R}^n) = \infty$, obviously.

3.ii) Properties of the Outer Measure

Prop 3.8 (Monotonicity) If $E_1 \subset E_2$, then $m^*(E_1) \leq m^*(E_2)$.

Proof : Coverings of E_2 also cover E_1.

Prop 3.9 (countable sub-additivity) If $E = \bigcup_{j=1}^{\infty} E_j$, then $m^*(E) \leq \sum_{j=1}^{\infty} m^*(E_j)$.

Proof : Can show for $m^*(E_j) < \infty$, else holds trivially.

Let Q_k^j be a collection of cubes covering E_j with

$$\sum_{k=1}^{\infty} |Q_k^j| \leq m^*(E_j) + \frac{\varepsilon}{2^j}$$

$$m^*(E) \leq \sum_{j,k=1}^{\infty} |Q_k^j| \leq \sum_{j=1}^{\infty}\sum_{k=1}^{\infty} |Q_k^j|$$

$$\leq \sum_{j=1}^{\infty} m^*(E_j) + \varepsilon \qquad \text{take } \lim \varepsilon \to 0 \quad \square.$$

Prop 3.10 : $m^*(E) = \inf_{\substack{U \subseteq \mathbb{R}^n \\ \text{open } E \subseteq U}} m^*(U)$.

Proof : $m^*(E) \leq m^*(U) \ \forall E \subseteq U$ by Prop 3.8. Let Q_j be a covering so

$$\sum |Q_j| \leq m^*(E) + \varepsilon/2.$$

Take U_j = open cube containing Q_j w/ area $\leq |Q_j| + \frac{\varepsilon}{2^{j+1}}$.

$$m^*(\cup U_j) \leq \sum m^*(U_j) \leq \sum |Q_j| + \frac{\varepsilon}{2^{j+1}} \leq \sum |Q_j| + \frac{\varepsilon}{2} \leq m^*(E) + \varepsilon \ \square$$

Prop 3.11 (Translation Invariance)
Suppose A is a set and $A+y=\{a+y \mid a\in A\}$ for a fixed $y\in\mathbb{R}^n$.
Then $m^*(A)=m^*(A+y)$.

Proof: If Q is a cube, then $Q+y$ is also, so a covering Q_j of A gives $m^*(A+y)\le\sum_{j=1}^{\infty}|Q_j+y| = \sum_{j=1}^{\infty}|Q_j|\le m^*(A)+\varepsilon$.
For reverse, note $(A+y)-y=A$. □

Corollary 3.12: If $E\subseteq\mathbb{R}^n$ is countable, $m^*(E)=0$.
Proof: $E=\bigcup_{j=1}^{\infty}E_j$ where $E_j=\text{pt}$. Follows from Prop 3.9.

3.iii) Measurable Sets

Def 3.13: A subset $E\subseteq\mathbb{R}^n$ is said to be measurable if for any $A\subseteq\mathbb{R}^n$,

$$m^*(A)=m^*(A\cap E)+m^*(A\cap E^c).$$

ie E cuts any A into two pieces. (Note $\le$ holds by subadditivity)

Ex 3.14: If $I\subseteq\mathbb{R}$ is an open interval, I is measurable.
Proof: Let $A\subseteq\mathbb{R}$ and let $A\subseteq\bigcup_{j=1}^{\infty}Q_j$ be a covering

Let $Q_j'=I\cap Q_j$, $Q_j''=I^c\cap Q_j$ $\Rightarrow$ Q_j' cover $A\cap I$, $Q_j''+\frac{\varepsilon}{2^j}$ cover $A\cap I^c$

$$m^*(I\cap A)+m^*(I^c\cap A)\le\sum_{j=1}^{\infty}|Q_j'|+\sum_{j=1}^{\infty}|Q_j''|+\varepsilon$$
$$=\sum_{j=1}^{\infty}|Q_j|+\varepsilon\le m^*(A)+2\varepsilon.$$

and now take $\lim \varepsilon\to 0$.

Ex 3.15: Same for closed intervals.
Ex 3.15.5: Same for $\mathbb{R}^n$ but subdivide

Prop 3.16 : If E_j are disjoint, measurable, then $\bigcup_{j=1}^{N} E_j$ [redacted]

[redacted] $m^*(A \cap \bigcup^N E_j) = \sum m^*(A \cap E_j)$. $\boxed{3.4}$

Proof : If $N=1$, this is vacuous. Take $A \subseteq \mathbb{R}^n$ then set $A' = A \cap (E_1 \cup E_2)$

$$m^*(A') = m^*(A' \cap E_1) + m^*(A' \cap E_1^c) \qquad (E_1 \text{ measurable})$$
$$= m^*(A \cap E_1) + m^*(A \cap E_2)$$

now use induction.

Thm 3.17 : The measurable sets form a σ-algebra, and
$$m^* : \mathcal{M}(\mathbb{R}^n) \to [0,\infty]$$
is a measure.

Proof : i) $\mathbb{R}^n$ is measurable, since $m^*(A) = m^*(A \cap \mathbb{R}^n) + \underbrace{0}_{= m^*(A \cap \emptyset)}$.

ii) E measurable $\Rightarrow$ E^c measurable by symmetry of def.

iii) Suppose E_i are measurable, then wts $\bigcup_{i=1}^{\infty} E_i$ is.

Step 1 : True for finite unions / intersections. Suppose E, F measurable

$$m^*(A) = m^*(A \cap E) + m^*(A \cap E^c)$$
$$= m^*(A \cap E \cap F) + m^*(A \cap E \cap F^c) + m^*(A \cap E^c \cap F) + m^*(A \cap E^c \cap F^c).$$
$$= m^*(A \cap E \cap F) + m^*(A \cap (E \cap F)^c)$$

$\bigcup = (E \cap F)^c$ by Prop 16.

Step 2 : Suppose E_i measurable, then
$$E_i' = E_i \setminus \bigcup_{j=1}^{i-1} E_j = E_i \cap \left(\bigcup_{j=1}^{i-1} E_j\right)^c$$
is measurable, and $\bigcup E_i' = \bigcup E_i$, so suffices to assume disjoint. Now let $F_N = \bigcup_{i=1}^{N} E_i$ (measurable)

$A \subseteq \mathbb{R}^n$ arbitrary.

$$m^*(A) = m^*(A \cap \bigcup^N E_i) + m^*(A \cap F_N^c)$$
$$= \sum_{i=1}^{N} m^*(E_i \cap A) + m^*(A \cap F_n^c)$$
$$\geq \sum_{i=1}^{N} \qquad\qquad + m^*(A \cap E^c) \quad \text{by monotonicity}$$

Take lim

$$\geq \sum_{i=1}^{\infty} m^*(E_i \cap A) + m^*(A \cap E^c)$$
$$\geq m^*(E \cap A) + m^*(E^c \cap A) \quad \text{by countable sub additivity.}$$

and $\mu(E) \geq 0$, $\mu(\emptyset) = 0$. If $E = \bigcup E_j$ disjoint,

$$\mu(E) \leq \sum_{j=1}^{\infty} m^*(E_j) \quad \text{(additivity)}.$$

~~$\sum_{j=1}^{\infty} m^*(E_j) \leq \sum_{j=1}^{\infty} \sum_{k} |Q_{jk}|$~~ covering

$\geq$ take $A = E$ in step 2.

<u>Def 3.18</u> : $\mathcal{M}(\mathbb{R}^n)$ are called the measurable sets, and $m^*|_{\mathcal{M}(\mathbb{R}^n)}$ is called the Lebesgue measure.

Lecture 4 — Properties of Lebesgue Measure + Counterexamples.

Recall last time we showed $(\mathbb{R}^n, \mathcal{M}(\mathbb{R}^n), m)$ is a measure space.

Thm 2.15: $(\mathbb{R}^n, \mathcal{M}(\mathbb{R}^n), m)$ is a measure space, and additionally satisfies

(1) Normalization $m(Q) = |Q|$

(2) Complete $E \in \mathcal{M}(\mathbb{R}^n)$ then $m(E) = 0 \Rightarrow A \subseteq E$ measurable w/ $m(A) = 0$.

(3) translation invariant $E \in \mathcal{M}(\mathbb{R}^n) \Rightarrow E+y \in \mathcal{M}(\mathbb{R}^n)$ w/ $m(E) = m(E+y)$

(4) Borel complete $\mathcal{B}(\mathbb{R}^n) \subseteq \mathcal{M}(\mathbb{R}^n)$.

Proof: (1) For open/closed Q, $m(Q) = m^*(Q) = |Q|$.

(2) Suppose E has outer measure 0. Then $A \cap E \subseteq E$, $A \cap E^c \subseteq A$ so by monotonicity

$$m^*(A) \geq m^*(A \cap E^c)$$
$$\geq m^*(A \cap E^c) + \overset{=0}{m^*(A \cap E)} \Rightarrow E \text{ is measurable.}$$

If $A \subseteq E$, then $m^*(A) = 0$ so also measurable.

(3) $A \cap (E + x) = (A - x) \cap E$, and m^* is translation inv.

(4) open sets $\subseteq \mathcal{M}(\mathbb{R}^n)$ and its a σ-algebra, □

4.i) Continuity of Measure

Prop 4.1: Suppose $E = \bigcap_{i=1}^{\infty} E_i$ with $E_1 \supseteq E_2 \supseteq \dots$ and $m(E_1) < \infty$.
Then $m(E) = \lim_{N \to \infty} m(E_N)$.

Proof: $E_1 = E \cup (E_1 \setminus E_2) \cup (E_2 \setminus E_3) \cup \dots$

$$m(E_1) = m(E) + \sum_{n=1}^{\infty} m(E_i \setminus E_{i+1}) = m(E) + m(E_1) - \lim_{i \to \infty} m(E_i)$$

$$\Rightarrow m(E) = \lim m(E_i)$$ □

Rem 4.2: Informally we can write

$$m(\lim E_i) = \lim m(E_i)$$

which looks more like standard continuity.

Lemma 4.2 (Borel - Cantelli) Suppose E_i measurable, $\sum_{n=1}^{\infty} m(E_n) < \infty$. [4.2]

Then $m\{x \mid x \in E_n \text{ for } \infty\text{-many } n\} = 0$.

Proof: Call this set A. $A \subseteq \bigcup_{n=N}^{\infty} E_n$ for any N.

$$m(A) \leq m\left(\bigcup_{n=N}^{\infty} E_n\right) \leq \sum_{n=N}^{\infty} m(E_n) \to 0$$

by convergence.

4.ii) Some funky sets.

Ex 4.3 (Cantor Set) $K \subseteq [0,1]$ a set uncountable but $m(K)=0$.

$K_0 = [0,1]$

$K_1 = K_0 \setminus \{\text{middle third}\}$

$K_2 = K_1 \setminus \{\text{middle thirds}\}$

$\vdots$

$K = \bigcap_{n=0}^{\infty} K_n$. Clearly $m(K_n) = \frac{2}{3} m(K_{n-1}) = \left(\frac{2}{3}\right)^n$.

$m(K) = \lim_{N\to\infty} m(K_N) = \lim \left(\frac{2}{3}\right)^N = 0$.

Claim: K is uncountable. (bijective w/ $\mathbb{R}$) $K = \{x \in [0,1] \mid x = 0.x_1x_2x_3\ldots$ in base 3 w/ $x_i = 0,2\}$.

Suppose we can enumerate $K_1, K_2, \ldots$ $\Rightarrow$ Cantor's diagonalization $\square$.

Corollary 4.4: There exist measurable sets that are not Borel.

In fact, "almost all" are such.

Proof: $m(K) = 0$, so any $A \subseteq K$ is measurable with measure 0.

$\Rightarrow P(K) \subseteq \mathcal{M}(\mathbb{R})$, but $P(K)$ has cardinality $2^{\mathbb{R}} \geq$

Borel sets have cardinality $\mathbb{R}$.

Ex 4.5 (Normal Numbers)

Def 4.6: $x \in [0,1]$ is normal if any finite sequence of digits occurs infinitely often.

Lemma 4.7 $m\{x \in [0,1] \mid x \text{ normal}\} = 1$.

Proof: Let $E_i = \{x \in [0,1] \mid i^{th} \text{ digit is } 1\}$. Clearly $m(E_i) = \frac{1}{10}$.

$$A = \{ 1 \text{ occurs inf often})$$
$$A^c = \{x \mid \exists N \text{ st } x \notin \bigcup_{n=N}^{\infty} E_n \} = \bigcup_{N=1}^{\infty} \bigcap_{n=N}^{\infty} E_n^c$$
$$m(A^c) = m\left(\bigcup_{N=1}^{\infty} \bigcap_{n=N}^{\infty} E_n^c\right) \leq \sum_{N=1}^{\infty} \underbrace{m\left(\bigcap_{n=N} E_n^c\right)}_{=0 \text{ by Borel-Cantelli.}} = 0.$$

$\Rightarrow m(A) = 1$.

$E_i^{\circ} = \{x \in [0,1] \mid i^{th} - i+p^{th} \ \_\ \_\ \_\ \_\ \}$ same proof.

Conjecture 4.8: $\pi, \sqrt{2}, -1$ are normal.

Corollary 4.9: It is not known if π contains Hamlet in some decimal form.

Ex 4.10: There exists a non-measurable set.

Let $\mathcal{N} = \{$ a lift of $\mathbb{R} \to \mathbb{R}/\mathbb{Q}$ ie a choice of representatives of cosets $x \sim y$ if $x - y \in \mathbb{Q}\}$. ← this invokes the axiom of choice!

Note each $x \in \mathbb{R}$ can be written $x = a + q$ uniquely w/ $a \in \mathcal{N}, q \in \mathbb{Q}$.

Lemma 4.11: $\mathcal{N}$ is not measurable.

Proof: Suppose it is. Set $S = [0,1] \cap \mathbb{Q}$.

$$B_{M,s} = \{s + a \mid s \in S,\ a \in \mathcal{N} \cap [M, M+1]\}$$

$\bigcup_{s \in S} B_{M,s} \subseteq [M, M+2]$, and all disjoint, and translates, ~~$\sum m(B_{M,s})$~~

Therefore

$$\sum_{s\in S} m(B_{M,s}) \leq 2 \implies m(B_{m,s}) = 0 \text{ b/c all equal.}$$

In particular, $m(\mathcal{N} \cap [m, m+1]) = m(B_{m,0}) = 0 \implies m(\mathcal{N}) = 0$.

But

$$\infty = m(\mathbb{R}) \overset{m}{=} \left(\bigsqcup_{q\in\mathbb{Q}} \mathcal{N}+q\right) = \sum 0 = 0 \quad \rightarrow\leftarrow \quad \square$$

<u>Corollary 4.12</u>: If $E \subseteq \mathbb{R}$ has positive measure, $\exists$ a $\mathcal{N} \subseteq E$ non-measurable.

<u>Remark 4.13</u>
<u>Thm 4.13</u> (Solovay 1970): A non-Lebesgue measurable set cannot be constructed without the Axiom of Choice.

<u>Rem 4.14</u>: If A is measurable $A - A = \{x - y \mid x, y \in A\}$ need not be measurable!

Note also $K - K = [-1, 1]$ despite $m(K) = 0$.

<u>Open Problem 4.15</u>: If $x \in K$ and $x \notin \mathbb{Q}$, then can x be algebraic?

Lecture 5 Measurable Functions

Def 5.1: A property P is said to hold "almost everywhere" if it holds on a set of full measure.

5.i) Littlewood's First Principle

Principle 5.2: "Every measurable set is closest to a finite union of intervals"

Def 5.3: Given two set A, B the symmetric difference

$$A \Delta B = (A \setminus B) \cup (B \setminus A)$$

Prop 5.4: If $m(E) < \infty$, then $\forall \varepsilon > 0$, $\exists J$ a finite union of intervals st

$$m(J \Delta E) < \varepsilon.$$

Proof: Note that $m(B \setminus A) = m(B) - m(A)$

since
$$m(B) = m(B \cap A) + m(B \cap A^c) = m(A) + m(B \setminus A)$$

Take $E \subseteq \bigcup Q_n$ such that

$$\sum |Q_n| \leq m(E) + \varepsilon/2.$$

Now take N large so that

$$\sum_{N+1}^{\infty} |Q_n| < \frac{\varepsilon}{2}.$$

$$m\left(E \setminus \bigcup^{N} Q_n\right) \leq m\left(\bigcup^{\infty} Q_n \setminus \bigcup^{N} Q_n\right) < \varepsilon/2$$

$$m\left(\bigcup^{N} Q_n \setminus E\right) \leq m\left(\bigcup^{\infty} Q_n \setminus E\right) \leq \varepsilon/2 \quad \square$$

Def 5.5: A G_δ set is a countable intersection of opens
an F_σ set is a countable union of closed

Note both are Borel.

Prop 5.6: A set $E \subseteq \mathbb{R}^n$ is measurable iff $\exists$

$$F_\sigma \subseteq E \subseteq G_\delta$$

such that $m(E \setminus F) = m(G \setminus E) = 0$.

Proof: If the statement holds, E is measurable since F, G are and all measure 0 sets are.

Let U_n be open with $m(U_n \setminus E_g) \leq \frac{1}{n}$, and set $G_\delta = \bigcap U_n$, and $m(G \setminus E) \leq \frac{1}{n} \; \forall n$.

Let V_n be open w/ $m(V_n \setminus E^c) \leq \frac{1}{n}$.

$$F_\sigma = \bigcup V_n^c$$

suffices. $\square$

5.ii) Measurable Functions (We will work only on $\mathbb{R}$ for now)

Prop 5.7: TFAE for $f: \mathbb{R} \to \mathbb{R}$

1) $f^{-1}(a, \infty]$ is measurable $\forall a \in \mathbb{R}$.
2) $f^{-1}(U)$ is measurable $\forall U \subseteq \mathbb{R}$ open
3) $f^{-1}(B)$ is measurable $\forall B \subseteq \mathbb{R}$ Borel.

Proof: $3 \Rightarrow 2 \Rightarrow 1$ a fortiori.

Let $\Sigma(f) = \{ A \in P(\mathbb{R}) \mid f^{-1}(A) \text{ is measurable} \}$

It is easy to check $\Sigma(f)$ is a σ-algebra, and (a, ∞) generates $\mathcal{B}$ so $\mathcal{B}(\mathbb{R}) \subseteq \Sigma(f)$

Prop 5.8: $\mathcal{M}(\mathbb{R})$ is an algebra containing $\mathcal{C}^0(\mathbb{R})$.

Proof: Obviously $\mathcal{C}^0 \subseteq \mathcal{M}$ by 2) of Prop 5.7.

1) $f \in \mathcal{M}(\mathbb{R})$ then $\alpha f \in \mathcal{M}(\mathbb{R})$ for $\alpha \in \mathbb{R}$ is clear, because $f^{-1}(a,b) = \alpha f^{-1}(\frac{a}{\alpha}, \frac{b}{\alpha})$

2) Suppose $f, g \in \mathcal{M}(\mathbb{R})$. Suppose $x \in (f+g)^{-1}(a,\infty)$.

Let $f(x) > \frac{p}{q} > a - g(x)$ be a rational, such a p/q exists iff $f(x)+g(x) > a$.

$$(f+g)^{-1}(a,\infty) = \bigcup_{p/q \in \mathbb{Q}} \{x \mid f(x) > p/q\} \cap \{x \mid \tfrac{p}{q} + g(x) > a\}$$

↑ countable union of measurable.

$\in \mathcal{M}(\mathbb{R})$.

3) If $f, g \in \mathcal{M}(\mathbb{R})$ then $fg \in \mathcal{M}(\mathbb{R})$.

$\frac{1}{2}[(f+g)^2 - f^2 - g^2] = fg$ so by 1), 2) enough to show for $f = g$.

$$(f^2)^{-1}(a,\infty) = f^{-1}(\sqrt{a},\infty) \cup f^{-1}(-\infty, -\sqrt{a}). \quad \square$$

Warning 5.9: $\mathcal{M}(\mathbb{R})$ is not closed under composition!

but $\mathcal{C}^0(\mathbb{R}) \times \mathcal{M}(\mathbb{R}) \longrightarrow \mathcal{M}(\mathbb{R})$

$(h, f) \longrightarrow h \circ f$ is measurable.

Theorem 5.10: Suppose f_n is a sequence of measurable functions such that $f_n \to f$ pointwise, then $f \in \mathcal{M}(\mathbb{R})$.

Proof: For $c \in \mathbb{R}$, write

$$f^{-1}(c,\infty) = \{x \mid \exists k \in \mathbb{N} \text{ st } \exists N \in \mathbb{N} \text{ st } \forall n \geq N,\ f_n(x) > c + \tfrac{1}{k}\}$$
$$= \bigcup_{k \in \mathbb{N}} \bigcup_{N \in \mathbb{N}} \bigcap_{n \geq N} \{x \mid f_n(x) > c + \tfrac{1}{k}\}.$$

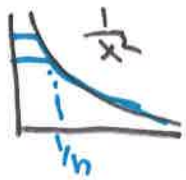
Countable $\bigcup$ and $\bigcap$ of measurable $\square$.

Def 5.11: The Indicator function of a measurable set A is
$$\chi_A = \begin{cases} 1 & x \in A \\ 0 & \text{else.} \end{cases}$$
It is clearly measurable.

Def 5.12: A function is called simple if it is a finite sum
$$f = \sum_{i=1}^{N} \alpha_i \chi_{E_i} \qquad E_i \text{ measurable.}$$
and a step function if E_i are disjoint intervals.

Def 5.13: f_n converges to f in measure if $\forall \varepsilon > 0$.
$$\lim_{n\to\infty} m\{x \mid |f_n(x) - f(x)| > \varepsilon\} = 0.$$

Ex 5.14:



$f_n = \min(n^2, \frac{1}{x^2})$ then $f_n \to f$ in measure but $\sup |f_n - f| = \infty$ so not uniformly.

Thm 5.15 Suppose $f: [a,b] \to \mathbb{R}$ is measurable. $\exists f_n \to f$ in measure such that f_n are
i) bounded and measurable
ii) Simple
iii) Step
iv) Continuous.

Proof i) let $f_n = \begin{cases} f(x) & |f(x)| \le n \\ 0 & \text{else.} \end{cases}$

then $|f_n - f| = 0$ outside $E_n = \{x \mid |f(x)| > n\}$.
Since $\bigcap E_n = \emptyset$ and $m(E_1) < \infty$, $m(E_n) \to 0$ by continuity of measure.

ii) By diagonalization can assume f bounded. Say $|f(x)| \le M$.
Let $I_{nk} = [-M + \frac{k}{n}, -M + \frac{(k+1)}{n}]$
$$E_k = \{x \mid f(x) \in I_{nk}\}$$
Take $f_n = \sum_{k=1}^{2Mn} (-M + \frac{k}{n}) \chi_{E_k}$. $|f_n - f| \le \frac{1}{n}$. Now send $n \to \infty$.

iii) Approximating χ_{E_k} by step is same as approximating E by opens.

iv)

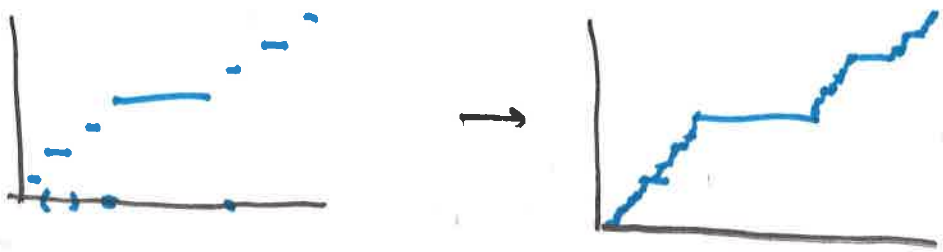
Lecture 6 Measurable Functions II: Littlewood's 2nd - 3rd Principles

6.i) The Cantor Function / Devil's Staircase

Ex 6.1: Let $C(x): [0,1] \to [0,1]$ be defined by

$$C(x) = \begin{cases} \frac{1}{2} & x \in (1/3, 2/3) \\ \frac{1}{4}, \frac{3}{4} & x \in (\frac{1}{9}, \frac{2}{9}) \text{ or } (\frac{7}{9}, \frac{8}{9}) \end{cases} \text{ on } K^c$$

$= \sup\{C(t) \mid t \in K^c_n \cap [0,x)\}$ on K



Lemma 6.2: $C(x): [0,1] \to [0,1]$ is continuous (hence measurable) and monotonically non-decreasing, but $C'(x) = 0$ a.e.

Proof: Clearly continuous on K^c. If $x \in K$, then x lies between two intervals of K^c_n w/ $K^c = \bigcup_n K^c_n$, for any n, and these have $|C(x) - C(y)| \leq \frac{1}{2^n}$.
Monotonically non-decreasing is obvious from sup.
$C'(x)$ exists and $= 0$ on K^c and $m(K^c) = 1$.

Corollary 6.3: (Even though it's not precisely defined yet)

$$\int_0^1 C'(x)\,dx = 0 \cdot m(K^c) + ? \cdot m(K) = 0$$

So in particular, the FTOC fails for C,

$$0 = \int_0^1 C'(x)dx \neq C(1) - C(0) = 1.$$

Corollary 6.4: $\exists \varphi$ a homeomorphism $\varphi: [0,1] \to [0,2]$ that sends a set of measure 0 to a set of positive measure, and vice-versa.

Proof: $\varphi(x) = x + C(x)$ is (strictly) monotonically increasing so a homeomorphism. $K \to$ measure 1.
$\frac{1}{2}\varphi^{-1}(2x)$ has opposite property.

6:ii) Littlewood's principles I

Principle 6.5 (Littlewood I: Measurable Sets): Any measurable set is nearly a finite union of open intervals

Principle 6.6 (Littlewood II: Measurable Functions): Any measurable function is nearly continuous

Principle 6.7 (Littlewood III: Convergence): Any convergent sequence is nearly uniformly convergent.

Here's a formal version of II:

Thm 6.8 (Lusin) Let $f : \mathbb{R} \to \mathbb{R}$ be measurable. Then given $\varepsilon > 0$, $\exists g : \mathbb{R} \to \mathbb{R}$ continuous s.t. $f \equiv g$ outside a set of measure $< \varepsilon$.

Recall we can construct $f_n \to f$ where $f : [a,b] \to \mathbb{R}$, f_n bounded, simple, step, C^0.

Proof: 1) It suffices to consider $f : [n, n+1] \to \mathbb{R}$, then use the "$\frac{\varepsilon}{2^n}$" trick.

2) Since $f_n \to f$ in measure, f_n bounded, can assume f bounded.

3) $g_n \in C^0$ st $g_n \to f$ in measure

$\Rightarrow \exists g_1$ st $m(\underbrace{|g_n - f|}_{h_1} > \frac{1}{2}) < \frac{\varepsilon}{2}$.

$$f = g_1 + h_1$$

Now $\exists g_{2m} \to g_1$ in measure $\Rightarrow \exists g_2$ st $m(|g_2 - g_1| > \frac{1}{2^2}) < \frac{\varepsilon}{4}$

Inductively

$$f = g_1 + \cdots g_N + h_N \text{ st}$$

g_j continuous, $|h_N| \leq \frac{1}{2^n}$ except on $E = \bigcup_{j=1}^{N} E_n$ with $m(E_n) \leq \frac{\varepsilon}{2^n}$, $h_N = 0$ on E^c.

4) Let $\sup |g_n(x)| \leq \sup |h_{N-1}| \leq \frac{1}{2^{n-1}}$, $g = \sum g$ converges uniformly and $f = g$ outside $m(\bigcup E_n) \geq \varepsilon$.

6.iii) Convergence

There are three types of convergence discussed so far.

$$\left\{\begin{matrix} f_n \to f \\ \text{in measure} \end{matrix}\right\} \overset{\text{implies}}{\subseteq} \left\{\begin{matrix} f_n \to f \\ \text{pointwise} \\ \text{(a.e.)} \end{matrix}\right\} \overset{\text{implies}}{\subseteq} \left\{\begin{matrix} f_n \to f \\ \text{uniformly} \end{matrix}\right\}$$

$m(|f_n - f| > \varepsilon) \to 0 \quad \forall \varepsilon$; $f_n(x) \to f \quad \forall x$; $\sup_x |f_n(x) - f(x)| \to 0$ = Convergence in C^0.

Prop 6.9: Suppose $f_n \to f$ pointwise a.e on $[a,b]$ then $f_n \to f$ in measure

Proof: Let $\varepsilon > 0$. $E_n = \{|f_n - f| > \varepsilon\}$

$B_N = \bigcup_{N=n}^{\infty} E_n$ so $B_N \supseteq B_{N+1} \supseteq \ldots$

$$\lim_{n\to\infty} m(E_n) \le \lim_{N\to\infty} m(B_N) \overset{\text{cont. of measure}}{=} m\left(\bigcap_{N=1}^{\infty} B_N\right)$$

$= m\{x \mid \forall N,\ \exists n \ge N \text{ w/ } |f_n - f| > \varepsilon\}$

$= 0$ by pointwise a.e. ∎

Ex 6.10: Note convergence $f_n \to f$ in measure for f_n bounded cannot happen on $\mathbb{R}$ (on any inf. measure set)

$f(x) = x$ is a counter example, However, any such f is still a lim in measure of f cont.

Ex 6.11: Prop 6.9 also fails on $\mathbb{R}$ b/c $m(B_1) = \infty$ is possible

$f_n = \sin x\ \chi_{[-N,N]} \to \sin x$ pointwise but

$m(|f_n - f| > \frac{1}{2}) = \infty \quad \forall n.$

Ex 6.11: $\exists f_n \to f$ in measure but not pointwise.

Let I_N = interval of length $\frac{1}{\sqrt{n}}$.

"Blimp function"

so $\bigcup_{N=n}^{\infty} I_n = [0,1]\ \forall N$, so $\sum |I_n| = \infty$.

Then $f_n = \chi_{I_n} \to 0$ in measure, but $f_n \to 0$ pointwise nowhere.

Thm 6.12: $f_n \to f$ in measure on $\mathbb{R}$, $\exists$ a subseq converging pointwise a.e.

Proof: $E_n = \{|f_n - f| > \frac{1}{2^n}\}$, take n_k such that $\{ m(E_{n_k}) < \frac{1}{k^2}$. Then $f_n \to f$ outside $\bigcup_{k=1}^{\infty} E_{n_k}$

but this $\to 0$ w/ k because $\sum m(E_{n_k}) < \infty$.

6.iv: Littlewood's 3rd Principle

Precise version:

Thm 6.13 (Egoroff): Let $f_n \to f$ pointwise a.e. on $[a,b]$. $\forall \varepsilon > 0$, $\exists E_\varepsilon \subseteq [a,b]$ s.t. $m([a,b] \setminus E_\varepsilon) < \varepsilon$ and $f_n \to f$ uniformly on E_ε.

Proof: Set $\delta_n = \sup_{i \geq n} |f_i(x) - f(x)|$

Note $\delta_n \geq \delta_{n+1}$ and $\delta_n \to 0$ pointwise a.e.

$\exists n_k$ s.t. $\delta_{n_k} \to 0$ in measure, so $\forall k$ $\exists$

E_k s.t. $m(E_k) \leq \frac{\varepsilon}{2^k}$ and $|\delta_{n_k}| \leq \frac{1}{k}$ on E_k^c.

Then $\delta_{n_k} \to 0$ uniformly on $E_\varepsilon := \bigcap E_k^c$ and

$m(E_\varepsilon) \geq m([a,b]) \setminus \sum E_k \geq (b-a) - \sum \frac{\varepsilon}{2^k} \geq b - a - \varepsilon$.

But δ_{n_k} decreasing so conv on subseq $\Rightarrow$ conv of whole seq $\square$

Lecture 7 | Lebesgue Integration I: bounded functions

Assumption 7.1: Throughout this lecture, assume E measurable w/ $m(E)<\infty$ and all measurable functions lie in

$$\mathcal{M}_B(E) = \{ f \in \mathcal{M}(E) \mid \exists M \text{ w/ } |f| \le M \}.$$

Def 7.2: Define the (Lebesgue) Integral of a function $f \in \mathcal{M}_B(E)$ w/ $m(E)<\infty$, denote

$$\int_E \; dx : \mathcal{M}_B(E) \longrightarrow \overline{\mathbb{R}}$$

or $\int_E \; dm(x)$ ← Lebesgue measure.

By

$$\int_E f\, dx := \sup_{\substack{\varphi \text{ simple} \\ \varphi \le f}} \left\{ \sum_{i=1}^N a_i m(E) \;\middle|\; \varphi = \sum_{i=1}^N a_i \chi_{E_i} \right\}.$$

Def 7.3: $f \in \mathcal{M}_B(E)$ is said to be (absolutely) Lebesgue integrable if

$$\int_E |f|\, dx < \infty$$

Rem 7.4: If $E' \subseteq E$, then one can extend

$$\overline{} : \mathcal{M}_B(E') \longrightarrow \mathcal{M}_B(E)$$

$$f \longmapsto \bar{f} = \begin{cases} f & x \in E' \\ 0 & \text{else} \end{cases}$$

and

$$\int_{E'} f\, dx = \int_E \bar{f}\, dx.$$

Prop 7.5: $\int_E \; dx : \mathcal{M}_B(E) \longrightarrow \overline{\mathbb{R}}$ satisfies

1) $\int_E af + bg\, dx = a\int_E f\, dx + b\int_E g\, dx$ (linearity)
2) $\int_E \chi_A\, dx = m(A)$ for $A \subseteq E$ measurable (normalization)
3) $f \le g \Rightarrow \int_E f\, dx \le \int_E g\, dx$ (monotonicity)
4) $\left| \int_E f(x)\, dx \right| \le \int_E |f(x)|\, dx$
5) $\int_E |f(x)|\, dx \le m(E) \sup |f(x)| < \infty$.

(7.2)

Rem 7.6: A simple function $\varphi : E \to \mathbb{R}$ has a canonical representation

$$\varphi = \sum_{k=1}^{N} \lambda_k \chi_{E_k}$$

where $E_k = \varphi^{-1}(\lambda_k)$ for all k. Other representations arise from subdividing or repetition. We assume all simple functions are in this canonical representation. (See Royden 4.2.1)

Prop 7.7: If $f \in \mathcal{M}_B(E)$ then

Note $\int_E \varphi \, dx = \Sigma' a_k m(E_k)$ is immediate →

$$\sup_{\varphi \leq f} \int_E \varphi \, dx = \int_E f \, dx = \inf_{\psi \geq f} \int_E \psi \, dx$$

agree and are finite. In fact, if f bounded, this is iff.

Proof: $\Rightarrow$ For any choice $\varphi \leq f \leq \psi$, so suffices to show $|\inf - \sup| < \varepsilon$ for any ε. Given ε, let $|f| \leq M$, and divide $[-M, M] = \bigsqcup I_n = \bigsqcup [a_n, b_n]$ where $|I_n| < \varepsilon$.

Set $E_n = f^{-1}(I_n)$. Then

$$\sum_{n=1}^{N} a_n m(E_n) \leq \sup_{\varphi \leq f} \Sigma' \alpha_k m(E_k) \quad \text{and on the other hand}$$

$$\inf_{\psi \geq f} \Sigma' \beta_k m(E_k) \leq \sum_{n=1}^{N} b_n m(E_n) \leq m(E) \cdot \varepsilon + \sum_{n=1}^{N} a_n m(E_n)$$

So $|\sup - \inf| \leq m(E) \cdot \varepsilon$.

$\Leftarrow$ Suppose $\sup = \inf$. Let φ_n, ψ_n be with $\frac{1}{2n}$ of these so

$$\int_E |\varphi_n - \psi_n| \, dx \leq \frac{1}{n}.$$

Since $\varphi_n \leq f \leq \psi_n$, $\forall n$ $\underbrace{\lim \varphi_n}_{:= \varphi} \leq f \leq \underbrace{\lim \psi_n}_{:= \psi}$ (monotonic, bounded above)

Thus $\varphi = \psi$ a.e. $\Rightarrow$ $f = \varphi = \psi$ a.e. $\Rightarrow$ f measurable since φ, ψ are.

Claim: $\varphi_n \to \psi_n$ in measure. ie $\forall \varepsilon > 0$ $\lim |m(|\varphi_n - \psi_n|_\varepsilon)| = 0$.

Pick $\delta > 0$. N st $\frac{1}{N} < \varepsilon \delta$. Then

$$m(|\varphi_n - \psi_n| > \varepsilon) \cdot \varepsilon \leq \int_E |\varphi_n - \psi_n| \leq \frac{1}{N} \leq \varepsilon \delta$$

$$m(\qquad) \leq \delta$$

(now subseq conv. but monotone) □

Proof (of Prop 7.5).

1) $\int_E f\,dx + \int_E g\,dx \leq \sup_{\varphi \leq f, \psi \leq g} \int \varphi + \psi \leq \sup_{\xi \leq f+g} \int_E \xi\,dx$

$= \int_E f + g.$

Using inf get reverse.

$\int_E \alpha f\,dx = \alpha \int_E f\,dx$ is obvious.

2) If $f = \chi_A$ then $\varphi \leq \chi_A \Rightarrow$ $\varphi \leq 0$ on A^c, $\varphi \leq 1$ on A, so

$\int_E \varphi\,dx \leq m(A)$, but $\varphi = \chi_A$ itself gives $=$.

3) If $f \leq g$, $\sup_{\varphi \leq f} \leq \sup_{\varphi \leq g}$ is the sup over a strictly larger set.

4) $-|f| \leq f \leq f$ so by 3, $-\int_E |f|\,dx \leq \int_E f\,dx \leq \int_E |f|\,dx$.

5) by inf def, $\inf \leq m(E) \cdot \sup |f|$.

$\sup \geq m(E) \cdot (-\sup |f|)$

Prop 7.8: If $A, B \subseteq E$ disjoint, measurable

$$\int_{A \cup B} f\,dx = \int_A f\,dx + \int_B f\,dx$$

Proof: By linearity

$$\int_{A \cup B} f\,dx = \int_E f \chi_{A \cup B}\,dx = \int_E f(\chi_A + \chi_B)$$

$$= \int_E f \chi_A\,dx + \int_E f \chi_B\,dx = \int_A f\,dx + \int_B f\,dx.$$

Thm 7.9 (Bounded convergence) Suppose $f_n \in M_B(E)$ are uniformly bounded $|f_n| \le M$, $f_n \to f$ pointwise a.e. Then

$$\lim_{n\to\infty} \int_E f_n\,dx = \int_E \lim f_n\,dx = \int_E f\,dx$$

Proof: By Littlewood III, $\exists A \subseteq E$ with $m(E \setminus A) < \varepsilon/2m$ st

$$\sup_{x\in A} |f_n(x) - f| \longrightarrow 0.$$

Now let $\varepsilon > 0$. n large enough so [that the sup above is] $\le \frac{\varepsilon}{2} \cdot m(E)$.

$$\left| \int_E f\,dx - \int_E f_n\,dx \right| \le \left| \int_A f_n - f\,dx + \int_{A^c} f_n - f \right|$$

$$\le \int_A |f_n - f|\,dx + \int_{A^c} |f_n - f|\,dx$$

$$\le m(A) \sup_{x\in A} |f_n - f| + m(A^c)\, 2m$$

$$\le m(E) \cdot \frac{\varepsilon}{2} m(E) + \frac{\varepsilon}{4m} \cdot 2m \le \varepsilon.$$

7.ii) Riemann Integrals

Def 7.10: Recall $f : [a,b] \to \mathbb{R}$ is Riemann integrable if

$$\lim_{n\to\infty} \sum_{n=1}^{N} \frac{b-a}{n} \cdot \inf_{x\in I_n} f = \lim_{n\to\infty} \sum_{n=1}^{\infty} \frac{b-a}{n} \sup_{x\in I_n} f$$

where $I_n = \left[a + \frac{b-a}{N}(n-1),\ a + \frac{b-a}{N} \cdot n\right]$.

Thm 7.11: If f is Riemann integrable, then it is Lebesgue integrable, and they agree.

Proof: $\frac{b-a}{n} \cdot \inf f = m(I_n) \cdot \inf_{I_n} f$ is a special case of a simple function, likewise for sup $\square$.

Lecture 8 | Lebesgue Integration II: functions on $\mathbb{R}$.

Throughout this lecture we assume $E = \mathbb{R}$ or has (possibly) infinite measure, and $f \in \mathcal{M}_+(E)$ is non-negative (at first) but not (necessarily) bounded (above).

8.i) Non-Negative functions

Def 8.2: The <u>support</u> of a function $f \in \mathcal{M}(\mathbb{R})$ is

$$\operatorname{supp} \varphi = \{x \mid \varphi(x) \neq 0\}$$

which is measurable. It has finite support if $m(\operatorname{supp} \varphi) < \infty$.

Recall if $f \in \mathcal{M}(E)$ then it may be extended by 0: f "=" $f\chi_E$.

Def 8.3: Define the <u>Lebesgue Integral</u>

$$\int_E dx : \mathcal{M}_+(E) \longrightarrow \overline{\mathbb{R}}$$

by

$$\int_E f\,dx = \sup_h \left\{ \int_E h \;\middle|\; \begin{array}{l} h \in \mathcal{M}_B(E) \text{ has finite support,} \\ 0 \leq h \leq f \end{array} \right\}.$$

Rem 8.4: By the way, $f \in \mathcal{M}(E)$ is tacitly allowed to take value $\pm\infty$ on a set of measure 0, thus e.g. $\frac{1}{x} \in \mathcal{M}(\mathbb{R})$ while technically perhaps we should write $\frac{1}{x} \in \mathcal{M}(\mathbb{R} \setminus \{0\})$.

Prop 8.5: $\int : \mathcal{M}_+(E) \to \overline{\mathbb{R}}$ still satisfies

1) Linearity
2) $\int_E \chi_A\,dx = m(A)$
3) $f \leq g \Rightarrow \int_E f\,dx \leq \int_E g\,dx$
4) $\left|\int_E f(x)\,dx\right| \leq \int_E |f(x)|$.

Proof: 3), 4) same as before. 2) same for $m(A) < \infty$, $m(A) = \infty$: $\sup \to \infty$ is obvious.

1) $\int f + \int g \leq \int f + g$ as before

$\geq$ used inf characterization, which we no longer have.

Let $h \leq f+g$ be bounded w/ finite support. Write

$h = h_1 + h_2$ w/ $h_1 = \min(h, f)$ $h_2 = h - h_1$.

Note that $h_1 \leq f$ by construction
$h_2 \leq g$ (if $h_1 = h$, then $h_2 = 0 \leq g$; $h_1 = f$, then $h_2 = h - h_1 \leq f + g - h_1 \leq g$)

Thus

$$\int_E h\,dx = \int_E h_1\,dx + \int_E h_2\,dx \quad \text{(by linearity for bounded, finite supp)}$$
$$\leq \int_E f\,dx + \int_E g\,dx$$

$\Rightarrow$

$$\sup \int_E h\,dx =: \int_E f+g\,dx \leq \int_E f\,dx + \int_E g\,dx$$

$\int_E \alpha f\,dx = \alpha \int_E f\,dx$ still immediate □

<u>Prop 8.6</u>: If A, B disjoint

$$\int_{A\cup B} f\,dx = \int_A f\,dx + \int_B f\,dx$$

continues to hold

<u>Proof</u>:

$$\int_{A\cup B} f\,dx = \sup_{h \text{ bd f.s.}} \int_{A\cup B} h = \sup_{h \text{ bd f.s.}} \int_A h\,dx + \int_B h\,dx$$
$$\leq \int_A f\,dx + \int_B f\,dx$$

but also $\geq$ up to ε by taking $h = h_1\chi_A + h_2\chi_B$ for h_i within $\varepsilon/2$. □

<u>8.ii) General (measurable) Functions</u>: Now drop the assumption $f \geq 0$.

<u>Def 8.7</u>: Define $f \in M(E)$ as <u>Lebesgue integrable</u> if

$$\int_E |f|\,dx < \infty.$$

Set $f^{\pm} = \max\{0, \pm f\}$, both measurable, and $\int_{\{f\geq 0\}} f^+\,dx + \int_{\{f\leq 0\}} f^-\,dx = \int |f| < \infty$
so both are integrable.

<u>Def 8.8</u>: If $f \in M(E)$ is Lebesgue integrable, define

$$\int_E f\,dx = \int_E f^+\,dx - \int_E f^-\,dx.$$

Prop 8.7 : Properties (1) - (4) continue to hold.

2) is same since $\chi_A \geq 0$.

3) Follows from linearity because $f \leq g \Rightarrow g - f \geq 0$ so $\int g - \int f \geq 0$.

4) Follows from 3.

1) Let $f, g \in \mathcal{M}(\mathbb{R})$. Write

$$f = f^+ - f^-$$
$$g = g^+ - g^- \qquad \text{(all are positive)}$$
$$f + g = h^+ - h^-$$

$$h^+ - h^- = f + g = f^+ - f^- + g^+ - g^-$$
$$h^+ + f^- + g^- = f^+ + g^+ + h^-$$

By linearity for positive functions

$$\int h^+ \, dx + \int f^- \, dx + \int g^- \, dx = \int f^+ \, dx + \int g^+ \, dx + \int h^- \, dx$$

$$\int f + g = \int h^+ - \int h^- = \int f^+ - \int f^- + \int g^+ - \int g^-$$
$$= \int f + \int g.$$

Scalar mult again clean.

8.iii) Some Inequalities : Assume $f \geq 0$.

Lemma 8.9 : Let $L_{f,t} = \{x \in E \mid f(x) \geq t\}$, $t \geq 0$.

$$\chi_{L_{f,t}} = \begin{cases} 1 & x \in L_{f,t} \\ 0 & \text{else.} \end{cases}$$

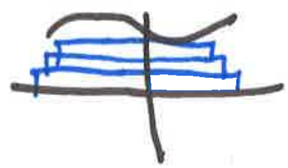
Then
$$f(x) = \int_0^\infty \chi_{L_{f,t}}(x)\, dt.$$

Proof : Observe

$$\chi_{L_{f,t}}(x) = \begin{cases} 1 & \text{if } f(x) \geq t \\ 0 & \text{if } f(x) < t \end{cases} = \chi_{[0,f(x)]}(t).$$

$$\int_0^\infty \chi_{L_{f,t}}(x)\, dt = \int_0^\infty \chi_{[0,f(x)]}(t) dx = \int_0^{f(x)} 1 \, dt = f(x).$$

Corollary 8.10 (Layer Cake Representation)



$$\int_0^\infty f\, dx = \int_0^\infty m(\{x \mid |f(x)| \geq t\}) dt$$

Proof :

$$\begin{aligned}
\int_0^\infty f(x) dx &= \int_0^\infty \int_0^\infty \chi_{[0,f(x)]}(t) dt\, dx \\
&= \int_0^\infty \int_0^\infty \chi_{L_{f,t}}(x)\, dt\, dx \\
&= \int_0^\infty \int_0^\infty \chi_{L_{f,t}}(x) dx\, dt \\
&= \int_0^\infty m(L_{f,t})\, dt
\end{aligned}$$

Pretend we can switch order, we will prove this later.

□

Thm 8.11 : (Markov's Inequality) If $f \geq 0$,

$$m(\{x \mid f \geq a\}) \leq \frac{\int_E f\, dx}{a}$$

Proof : $a \cdot m(\ ) \leq \int f(x) dx$ is obvious.

Thm 8.12 (Chebychev's Inequality) If $\int |f|^p dx < \infty$ for $p > 0$,

$$m(\{x \mid f \geq a\}) \leq \frac{1}{a^p} \int_E |f|^p dx$$

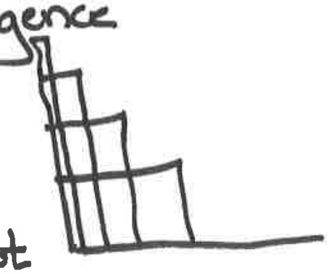
Proof : Markov applied to $|f|^p$ □

Lecture 9 Lebesgue Integration III: the convergence theorems.

Question 9.1: Suppose $f_n \to f$ pointwise a.e. When can we conclude
$$\lim_{n\to\infty} \int_E f_n \, dx = \int_E \lim_{n\to\infty} f_n \, dx \ ?$$

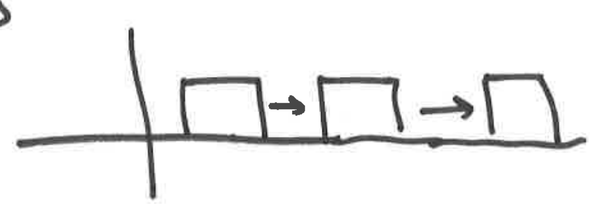
Ex 9.2: Two phenomena can ruin convergence

1) Mass escapes vertically
$$f_n = n \cdot \chi_{[0,1/n]}$$



Then $f_n \to 0$ pointwise a.e., but
$$\lim_{n\to\infty} \int_0^1 f_n \, dx = \lim_{n\to\infty} 1 = 1 \neq 0 = \int_0^1 f \, dx.$$

2) Mass escapes horizontally
$$f_n = \chi_{[n,n+1]}$$



Then $f_n \to 0$ pointwise,
$$\lim_{n\to\infty} \int_{\mathbb{R}} f_n \, dx = \lim_{n\to\infty} 1 \neq 0 = \int_{\mathbb{R}} f \, dx.$$

Rem 9.3: Recall convergence holds if $m(E) < \infty$ and $|f_n| \leq M$ as it rules out both these behaviors, i.e. Ex 9.2 is "all" that can go wrong.

9.i Fatou's Lemma

Lemma 9.4 (Fatou's Lemma) Suppose $f_n \geq 0$ are non-negative measurable functions on E. If $f_n \to f$ pointwise a.e,
$$\int_E f \leq \liminf_n \int_E f_n.$$

Proof: Note f is measurable since it is pointwise a.e. limit, $f \geq 0$.

Let h be bounded, finite support with $0 \leq h \leq f$.

Define $h_n = \min\{h, f_n\}$ on $E \supseteq \operatorname{supp} h$.

Then $0 \le h_n \le h \le M := \sup h$, so is bounded,

and $\operatorname{supp}(h_n) \subseteq \operatorname{supp} h$.

By Bounded Convergence Thm (7.9),

$$\lim_{n\to\infty} \int_E h_n = \lim_{n\to\infty} \int_{\operatorname{supp} h} h_n = \int_{\operatorname{supp} h} h = \int_E h.$$

But $h_n \le f_n$ $\forall n$ so

$$\int_E h = \lim_{n\to\infty} \int_E h_n \le \liminf_{n\to\infty} \int_E f_n$$

If $\liminf = \infty$ trivial, if $\liminf < \infty$ take N st $\inf_{n \ge N} = \operatorname{Inf} + \varepsilon/2$; increase n so that $\lim \int h_n$ within $\varepsilon/2$ ∎

Now take $\sup_{\substack{h \le f \\ \operatorname{supp} h \text{ finite}}}$.

Rem 9.5 : Ex 9.2 shows inequity may be strict w/ right side including the "escaped" mass.

9.ii Monotone Convergence:

Theorem 9.6 (Monotone Convergence) Suppose f_n is an increasing sequence of non-negative functions on E, and $f_n \to f$ pointwise a.e. Then

$$\lim_{n\to\infty} \int_E f_n \, dx = \int_E \lim_{n\to\infty} f_n \, dx.$$

Proof: By Fatou,

$$\int_E f \le \liminf_{n\to\infty} \int_E f_n \, dx$$

But

$$\le \limsup_{n\to\infty} \int_E f_n \, dx \le \limsup_{n\to\infty} \int_E f \, dx = \int_E f \, dx.$$

Therefore $\liminf = \limsup = \int_E f \, dx$. ▫

9.iii) Dominated Convergence

Def 9.7: A measurable function g is said to dominate a sequence f_n if $|f_n| \leq g$ on E for all n.

Lemma 9.8 (Modulus of Integrability) Suppose $f \geq 0$ is integrable. Then $\forall \varepsilon > 0$, $\exists \delta$ such that $m(A) < \delta \Rightarrow \int_A f\,dx < \varepsilon$.

Proof: Let $\varepsilon > 0$. For M sufficiently large, $\exists f_M$ with $|f_M| \leq M$ and

$$\int_E |f - f_M|\,dx < \frac{\varepsilon}{2}.$$

Now let $A \subseteq E$ be measurable w/ $m(A) < \delta := \frac{\varepsilon}{2M}$.

Then

$$\left|\int_A f\,dx\right| \leq \int_A |f|\,dx \leq \int_A |f - f_M + f_M|\,dx$$
$$\leq \int_A |f - f_M|\,dx + \int_A |f_M|\,dx$$
$$\leq \int_E |f - f_M|\,dx + m(A) \cdot M$$
$$\leq \frac{\varepsilon}{2} + \frac{\varepsilon}{2} \qquad \square$$

Corollary 9.9: $f : \mathbb{R} \to \mathbb{R}$ integrable, $F(x) = \int_{-\infty}^{x} f(t)\,dt$ is uniformly continuous.

Thm 9.10 (Lebesgue Dominated Convergence) Suppose $f_n \to f$ pointwise a.e. with $|f_n|, |f|$ dominated by g with $\int_E g\,dx < \infty$. Then $\lim_{n\to\infty} \int_E f_n\,dx = \int_E \lim_{n\to\infty} f_n\,dx$

Proof: By Lemma 9.8, given $\varepsilon > 0$ $\exists \delta$ such that $A \subseteq E$ with $m(A) < \delta$ $\Rightarrow \int_A g\,dx < \varepsilon/2$.

Also $\exists h \leq g$ supported on $[-m, m]$ st $\int g - h\,dx < \frac{\varepsilon}{2}$.

By Littlewood 3, $\exists A$ with $m(A) < \delta$ s.t. $f_n \to f$ uniformly outside A. $\subseteq [-m, m]$.

Then

$$\left|\int_E f_n - f\,dx\right| \le \int_E |f_n - f|\,dx$$

$$\le \int_{E \setminus [-m,m]\cap E} |f_n - f|\,dx + \int_{[-m,m]\cap E \setminus A} |f_n - f| + \int_A |f_n - f|\,dx$$

Note m is defined by g so ind. of n!

$$\le 2\int_{E\setminus[-m,m]\cap E} |g|\,dx + \int_{[-m,m]\cap E \setminus A} |f_n - f| + 2\int_A |g|\,dx$$

$$\le 2\cdot\frac{\varepsilon}{2} + 2m\cdot \sup_{\text{uniform}} |f_n - f| + 2\cdot\frac{\varepsilon}{2}$$

$$\le \varepsilon + \varepsilon \quad \text{for } n \text{ large} \quad \square.$$

9.iv) Examples

Ex 9.11: Evaluate $\lim_{k\to\infty}\int_0^1 \frac{k[\sin(kx) + \sin(\frac{1}{k}x)]}{\tan(\frac{1}{k})\cdot\frac{\pi}{2}x) + k^{3/2}\sqrt{x}}\,dx$.

$$\left|\frac{k[\sin(kx) + \sin(\frac{1}{k}x)]}{\tan(\frac{\pi}{2k}x) + k^{3/2}\sqrt{x}}\right| \le \frac{2k}{\text{positive} + k^{3/2}\sqrt{x}} \le \frac{1}{\sqrt{kx}} \le \frac{1}{\sqrt{x}}$$

and $\int_0^1 \frac{1}{\sqrt{x}}\,dx = 2x^{1/2}\Big]_0^1 = 2 < \infty$. So

$$\lim_{k\to\infty}\int_0^1 \underline{\qquad}\,dx = \int_0^1 \lim \underbrace{\qquad}_{=0} = 0.$$

Ex 9.12: Suppose $f(x)$ is C^1 a.e. and $|f'(x)| \le M$. Then wrt t

$$\frac{d}{dt}\int_{\mathbb{R}} f(t,x)\,dx = \int_{\mathbb{R}} f'(x,t)\,dx$$

Proof: $|f'|$ dominates the difference quotients $\frac{f(t+h) - f(t)}{h}$.

Corollary 9.13: If $E_1 \subseteq E_2 \subseteq \ldots$ $\int_{\cup E_n} f = \lim_{n\to\infty}\int_{E_n} f$ and $\int_E |f| < \infty$

$E_1 \supseteq E_2$ $\int_{\cap E_n} f = \lim_{n\to\infty}\int_{E_n} f$. (Apply LDC to $\chi_{E_n} f$.)

Lecture 10 Differentiation and Integration I: monotone functions

Question 10.1: Under what assumptions of f does the FTOC

$$\frac{d}{dx}\int_0^x f(t)dt = f(x)$$

hold? Note we saw that it fails for $C(x)$ the Cantor function.

Ex 10.2: There exists a continuous but nowhere differentiable function. Let $a_n = 10^{-n}$, $b_n = 10^{2n}$.

Consider

$$f = \sum_{n=1}^{\infty} a_n \sin(b_n x) \quad \text{on } [0, 2\pi].$$

f is a uniform limit of continuous, so continuous, but

$$f'(x) = \sum_{n=1}^{\infty} a_n b_n \sin(b_n x) = \sum_{n=1}^{\infty} 10^n \cos(b_n x).$$

would have to hold. Can show $\frac{f(x+h)-f(x)}{h} \to \infty$ everywhere.

10.i) Monotone Functions

Def 10.3: A function $f:[a,b] \to \mathbb{R}$ is said to be monotone if $x \leq y \Rightarrow f(x) \leq f(y)$.

Prop 10.4: Let $f:[a,b] \to \mathbb{R}$ be monotone. Then f is continuous away from a countable (hence measure 0) set.

Proof: Suffices to assume $[a,b]$ finite and take unions.

Let $f^-(x_0) = \sup\{f(x) \mid a < x < x_0\}$

$f^+(x_0) = \inf\{f(x) \mid x_0 < x < b\}$.

Discontinuity iff $f^- < f^+$ is strict. If so $[f^-(x_0), f^+(x_0)] \subseteq [f(a), f(b)]$.

$\forall n \in \mathbb{N}$, only finitely many have size $> \frac{1}{n}$. So

$$\text{discontinuities} = \bigcup_{n\in\mathbb{N}} \{\text{finite}\}.$$

□

Ex 10.5: If $C \subseteq [a,b]$ is countable $\exists f:[a,b] \to \mathbb{R}$ monotonic with discontinuities at C.

Enumerate $C = \{q_1, q_2 \dots\}$

$$f(x) = \sum_{i \text{ st } q_i < x} \frac{1}{2^i}.$$

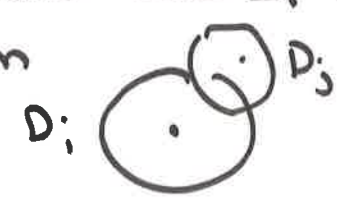
Thm 10.6 (Monotone Differentiability) If $f:[a,b] \longrightarrow \mathbb{R}$ is monotone it is differentiable almost everywhere.

10.ii) Covering Lemmas

Lemma 10.7: Let $K \subseteq \mathbb{R}^n$ be compact. If $\mathcal{U} = \bigcup_{i \in I} D_i$ is a cover by balls, $\exists$ $i = 1, \dots N$ s.t.

$$K \subseteq \bigcup_{i=1}^{N} 3D_i$$

Proof: Let D_i for $1 \dots N$ be a finite subcover ordered by decreasing radii. $D_i = B(x_i, r_i)$ $r_1 \geq r_2 \geq \dots$
Starting from $i=1$ throw out each D_i if it is not disjoint from those already selected. Suppose $x \in K$. Either is is in chosen D_i or another D_j. In the latter case $\exists_i$ so $x \in D_i \cap D_j$ w/ $i \geq j$, but then $x \in 3D_i$



$\square$

Def 10.8: A collection $\mathcal{U} = \{D_i\}$ of balls (radii r_i) is a Vitali covering of K if $\forall x, \forall \varepsilon > 0$, $\exists i \in I$ such that $x \in D_i$ and $r_i < \varepsilon$.

Lemma 10.9 (Vitali Covering Lemma) Let $E \subseteq \mathbb{R}$ be a set of finite positive measure, and let $\mathcal{U}$ be a Vitali covering. Then $\forall \varepsilon > 0$ $\exists$ finite collection $D_i \in \mathcal{U}$ s.t.

$$m\left(E \,\Delta \bigcup_{i=1}^{N} D_i\right) < \varepsilon.$$

Lemma 10.10: Let $\mathcal{U}$ be a Vitali covering of K compact. Then $\forall N \in \mathbb{N}$ $\exists D_i$ disjoint so

$$K \subseteq \bigcup_{i=1}^{N} \overline{D}_i \cup \bigcup_{i=N+1}^{\infty} 3D_i$$

Proof: Assume $\mathcal{U} = \bigcup_{n \in \mathbb{N}} B(x_i^n, \frac{1}{n})$ for x_i^n finite, by compactness. Proceed w/ algorithm as before on (now countable sequence). Either $x \in \bigcup_{i=1}^{N} \overline{D}_i$ or it lies in a ball thrown out, and radii $r_i \geq r_j$, so same logic. □

Proof (of Vitali Covering Lemma) We may choose

$$K \subseteq E \subseteq \mathcal{U}. \quad \text{and } m(\mathcal{U} \setminus K) < \varepsilon.$$

(K: closed, finite measure $\Rightarrow$ compact; $\mathcal{U}$: open)

By 10.10, $K \subseteq \bigcup_{i=1}^{N} \overline{D}_i \cup \bigcup_{N+1}^{\infty} 3D_i$ $\forall N$,

By throwing out balls of large radii, may assume each $D \in \mathcal{U}$ lies in $\mathcal{U}$.

Since D_i disjoint, $\sum^{\infty} m(D_i) < m(\mathcal{U}) < m(E) + \varepsilon$.

Let N be large enough so that $\sum_{N+1} 3m(D_i) < \varepsilon$.

$$m\left(E \setminus \bigcup_{i=1}^{N} D_i\right) \leq m\left(\mathcal{U} \setminus \bigcup_{i=1}^{N} D_i\right) \leq m\left(\mathcal{U} \setminus \left(K \cap \bigcup^{N} D\right) \cup \bigcup^{N} D \setminus K\right)$$

$$\leq m(\mathcal{U} \setminus K) + m\left(K \setminus K \cap \bigcup D_i\right)$$

$$\leq 2\varepsilon.$$

$$m\left(\bigcup_{i=1}^{N} D_i \setminus E\right) \leq \varepsilon + m\left(\bigcup_{i=1}^{N} D_i \setminus K\right)$$

$\leq 2\varepsilon$ since in $\mathcal{U}$. □

Proof (of monotone differentiability)

Suffices to assume $[a,b]$ finite, and f continuous (b/c discont are measure 0)

For $x \in [a,b]$, let $I_{\delta_1 \delta_2} = [x - \delta_1, x + \delta_2]$.

Set $$D_{\delta_1 \delta_2}(x) = \frac{f(x+\delta_2) - f(x-\delta_1)}{(\delta_2 - \delta_1)}.$$

Clearly $\lim_{\substack{\delta_1 \to 0 \\ \delta_2 \to 0}} D_{\delta_1 \delta_2}(x)$ exists $\Rightarrow$ $f'(x)$ exists ~~by monotonicity~~ by continuity.

Thus if f' does not exist $\lim_{\delta_1,\delta_2 \to 0} D_\delta(x)$ does not exist,
and $\exists$ intervals I, J so that

$$D_I(x) \neq D_J(x) \quad \Rightarrow \exists \text{ a rational in between.}$$

Let $E_{r,s}$ be $\{x \mid \exists r,s \in \mathbb{Q}$ w/ $D_I(x) < r < s < D_J(x)\}$.
$\exists I, J$ arbitrarily small.

Thus if $f'(x)$ DNE, $x \in E_{r,s}$ for some r,s.

<u>Claim</u>: $m(E_{r,s}) = 0$ (hence $m(\bigcup E_{r,s}) = m\{x \mid f' \text{ DNE}\} = 0$).

<u>Idea</u>: Intuitively, $f' > s$, and $f' < r$ both hold, so we show
$s \cdot m(A) \leq m(f(E)) \leq r\, m(A)$ → since $r \neq s$ if $m(A) \neq 0$.

Let $\varepsilon > 0$. By Vitali's Lemma $\exists \bigcup_{n=1}^{N} I_n$ (disjoint) st
$m(E_{r,s} \,\Delta\, \bigcup_{n=1}^{N} I_n) < \varepsilon$.

In fact, can assume $D_{I_n}(x) < r$. (Take $\mathcal{U}$ to be the covering of $E_{r,s}$ by the intervals I of size $\frac{1}{n}$).

Thus $\sum' |f(I_n)| \leq r \sum' |I_n|$

Now apply Vitali to $E_{rs} \cap \bigcup_{n=1}^{N} I_n$ using (disjoint) covering by Js.

$\exists \bigcup_{m=1}^{M} J_m$ st $m(E_{rs} \cap \bigcup_{n=1}^{N} I_n \,\Delta\, \bigcup_{m=1}^{M} J_m) < \varepsilon$ and $D_{J_n}(x) > s$.

$\Rightarrow \sum' |f(J_n)| \geq s \sum_{m=1}^{M}' |J_m|$.

→ monotonicity $\bigcup J_m \subseteq \bigcup I_n$ and disjoint

$s\, m(\bigcup J_m) \leq s \sum' |J_m| \leq \sum' |f(J_m)| \leq \sum' |f(I_n)| \leq r \sum' |I_n| = r\, m(\bigcup I_n)$

$|m(E_{rs}) - m(\bigcup I_n)| \leq \varepsilon$
$|m(E_{rs}) - m(\bigcup J_m)| \leq 2\varepsilon$. $\Rightarrow$ $\lim \varepsilon \to 0$
$s\, m(E_{rs}) \leq r\, m(E_{rs})$
→ ↯ unless $m(E_{rs}) = 0$

Lecture 11 Integration and Differentiation II: absolute continuity

Recall if $F:[a,b]\to\mathbb{R}$ is measurable and monotone, then $F'=f$ exists a.e.

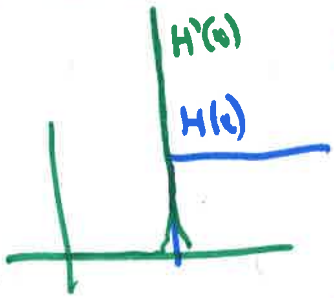
Ex 11.1: As before if $C(x):[0,1]\to\mathbb{R}$ is the Cantor function, $C'=f$ indeed exists and is 0 a.e.

Hence

$$0=\int_0^1 C'(x)\neq C(1)-C(0)=1.$$

But $\leq$ is true.

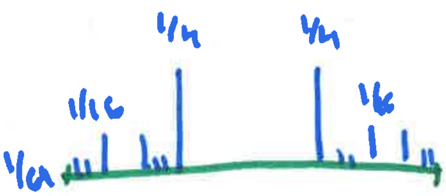
Intuition 11.2: Consider $H(x)=\begin{cases}0 & x\leq 1/2\\ 1 & x\geq 1/2\end{cases}$



Then intuitively $H'(x)=\delta(x-1/2)$ is the derivative

$$\int_a^b H'(x)=\begin{cases}1 & \text{if } a\leq 1/2\leq b\\ 0 & \text{else.}\end{cases}$$

For the Cantor function, increase comes from $\sum_{c\in C}\frac{1}{4^k}\delta_{c_k}$, k = layer of cantor set.



Thm 11.3: If $f:[a,b]\to\mathbb{R}$ is monotone increasing,

$$\int_a^b f'(x)\,dx\leq f(b)-f(a).$$

Proof: Let $f_n(x)=\frac{f(x+1/n)-f(x)}{1/n}$. ($\leftarrow$ extend f by $f(a)$, $f(b)$ const.) By f' existing a.e

$f_n(x)\to f'(x)$ pointwise a.e.

Thus

$$\int f'\leq\liminf\int f_n\leq\limsup\int f_n$$

$$\int_a^b f_n=\frac{1}{1/n}\int_{a+1/n}^{b+1/n}f(x)\,dx-\frac{1}{1/n}\int_a^b f(x)dx=\frac{1}{1/n}\int_b^{b+1/n}f(x)dx-\frac{1}{1/n}\int_a^{a+1/n}f(x)dx\leq f(b)-f(a)$$

(by monotone.) $\square$

11.ii) Absolute Continuity + Bounded Variation

Q 11.4: What is the condition for equality? We have to rule out the δ-function jumps. We will define

$$BV[a,b] \supseteq AC[a,b]$$

$$= \{f \text{ has bounded variation}\} \qquad = \{f \text{ absolutely continuous}\}$$

Def 11.5: A function has bounded variation if

$$\|f\|_{BV} := \sup_{p \in [a,b]} \sum_{i=0}^{n} |f(a_{i+1}) - f(a_i)|$$

is finite, where the sup is over all finite partitions

$$a = a_0 < a_1 < a_2 < \dots < a_n = b.$$

Prop 11.6: A function f has bounded variation iff

$$f = g(x) - h(x)$$

for g, h monotone increasing.

Proof: Suppose $f = g(x) - h(x)$. Then

$$\|f\|_{BV} = \sum_{k=0}^{n} |g(x_{k+1}) - g(x_k) + h(x_{k+1}) - h(x_k)|$$

$$\leq \sum_{k=0}^{n} |g(x_{k+1}) - g(x_k)| + |h(x_{k+1}) - h(x_k)|$$

$$\leq g(b) - g(a) + h(b) - h(a) < \infty$$

Now suppose $\|f\|_{BV} < \infty$. Write

$$y = y^+ - y^- \text{ for } y^\pm \geq 0.$$

$$f_\pm(x) = \sup_p \sum_{k=0}^{n} [f(a_{k+1}) - f(a_k)]_\pm$$

where p partitions $[a,x]$.

Then $f_\pm$ are monotone and bounded since $\|f\|_{BV} < \infty$.

Claim $f = f_+ - f_- + f(a)$.

Note $f_\pm$ are increasing as we refine partition (by triangle ineq)
so can find P subdivision for which both are within $\varepsilon/2$ of sup.

Then
$$\sum_{i=0} [f(a_{i+1}) - f(a_i)]_+ - [f(a_{i+1}) - f(a_i)]_-$$
$$= \sum_{i=1}^{n} f(a_{i+1}) - f(a_i) = f(a_n) - f(a_0) = f(x) - f(a) + \varepsilon.$$

__Corollary 11.7__: If $f \in BV[a,b]$ then f' exists a.e. and $f' \in L^1[a,b]$

__Proof__: apply monotone differentiability to g, h. □

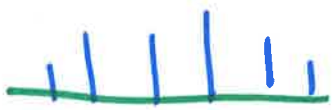
__Def 11.8__: A function $F : [a,b] \to \mathbb{R}$ is absolutely continuous if $\forall \varepsilon > 0$, $\exists \delta > 0$ st. for and $\sqcup I_n$ w/
$\sum |I_n| < \delta$,
$$\sum_{i=1}^{n} |f(a_i) - f(b_i)| < \varepsilon.$$
For $I_j = (a_j, b_j)$.

__Prop 11.9__: Suppose $f : [a,b] \to \mathbb{R}$ is integrable. Then
$F(x) = \int_a^x f\,dt$ is absolutely continuous.

__Proof__: $\forall \varepsilon > 0$ $\exists \delta$ s.t. if $\mu(A) < \delta$ then $\int_A |f|\,dx < \varepsilon$.

(approximate by bounded)

Ex 11.10: Absolute continuity is designed to precisely rule out the cantor function. If f has discontinuity jumps



then $\sqcup I_n$ can be taken $< \varepsilon$ while $|f(b)-f(a)| >$ const.

Eg. 2^n intervals of length 3^{-n} so that

$\sum |f(I_n)| = 1$ but $\sum |I_n| = (\frac{2}{3})^n \to 0$.

Prop 11.11: Recall Lipchitz means $\exists M$ st

$$|f(b)-f(a)| \leq M|b-a|.$$

One has (Lipschitz)

$$C^{0,1}[a,b] \subseteq AC[a,b] \subseteq BV[a,b]$$

Proof: Let $\varepsilon > 0$, and δ be given by A.C. Then

$$\|f\|_{BV} \leq \frac{\varepsilon (b-a)}{\delta}$$

For Lipschitz, take $\delta = \frac{\varepsilon}{M}$ □

Corollary 11.12: If $f \in AC[a,b]$ then f' exists a.e.

11.iii) The FTOC

Theorem 11.13 (FTOC with absolute continuity)

1) Suppose $f \in L^1[a,b]$. Then $F = \int_a^x f \, dt$ is A.C. and

$$\frac{d}{dx} F = f \quad \text{(a.e. hence in } L^1\text{)}.$$

2) Suppose $f \in AC[a,b]$. Then

$$\int_a^x f'(t)dt = f(x) - f(a).$$

Proof (next time)

Lecture 12 | Integration and Differentiation III: generalized FTOC.

Recall we wish to show

Thm 11.13 (generalized FToC)

1) If $f \in L^1[a,b]$, then $F(x) = \int_a^x f\,dt \in AC[a,b]$ and $F'(x) = f$ a.e.

2) If $f \in AC[a,b]$, then f' exists a.e. and
$$\int_a^x f'(t)\,dt = f(x) - f(a).$$

Rem 12.1: Phrased a different way,

$$AC[a,b]/_{\mathbb{R}} \overset{D=\frac{d}{dx}}{\simeq} L^1[a,b]$$

$$I = \int_a^x dt$$

is an isomorphism

12.i) The lemmas

Intuition 12.2: We will show $D \circ I$ and $I \circ D$ give the identity.

$\Rightarrow$ $D \circ I$ first prove for Lipschitz, then take limits

$I \circ D$ shown to be injective.

Lemma 12.3: If $I(f) = 0$ for $f \in L^1[a,b]$, then $f = 0$, ie. I is injective.

Proof: If $f \neq 0$ in L^1, then $\exists E \subseteq [a,b]$ s.t. $f \neq 0$ on E w/ $m(E) \geq 0$. Take $F \subseteq E$ closed of positive measure. $\mathring{F}$ is a union of intervals, take one, J. $\int_J f \neq 0$ so $J = [c,d]$ cannot have $\int_a^c f\,dt = \int_a^d f\,dt = 0$ $\square$

Lemma 12.4 (Lipschitz Case) Suppose $f \in L^\infty[a,b]$. Then $I(f)$ is Lipschitz, and $D \circ I(f) = f$ a.e.

Rem 12.5: This shows the theorem on the subspaces

$$\begin{array}{ccc} C^{0,1}[a,b] & \simeq & L^\infty[a,b] \\ \cap & & \cap \\ AC[a,b] & \simeq & L^1[a,b] \end{array}$$

Proof: Let $|f| \leq M = \|f\|_{L^\infty}$. Then $F = I(f)$ satisfies

$$|F(x+h) - F(x)| = \left| \int_x^{x+h} f(t)\,dt \right| \leq M \cdot h$$

is Lipschitz. Let $F_n = \frac{1}{1/n}[F(x+1/n) - F(x)] \leq M$ and $F_n \to F'$ pointwise a.e. so by dominated convergence.

$$\int_a^x F'(t)\,dt = \lim_{n\to\infty} \int_a^x n[F(t+1/n) - F(t)]\,dt$$

$$= \lim_{n\to\infty} \int_{a+1/n}^{x+1/n} nF(t)\,dt - \int_a^{x} nF(t)\,dt$$

$$= \lim_{n\to\infty} \int_x^{x+1/n} nF(t) - \int_a^{a+1/n} nF(t)\,dt = F(x) - F(a)$$

$$= \int_a^x f(t)\,dt$$

Thus $\int_a^x F'(t) - f(t)\,dt = 0 \;\forall x$, so $F' = f$ a.e. by 12.3.

Lemma 12.6: Let X, Y be sets and $X \underset{I}{\overset{D}{\rightleftarrows}} Y$ maps such that $D \circ I = id_Y$. If D is injective, then $I \circ D = id_X$.

Proof: $D(I(Dx)) = Dx$, hence $I \circ Dx = x$ by injectivity.

Proof $I \circ D = id$ (general case) Suppose $f \in L^1[a,b]$

Write $f = f^+ - f^-$, so enough to assume $f \geq 0$.

Let $f_n = \min(n, f)$. hence $f_n \to f$ pointwise and is monotone.

Also $F_n = I(f_n)$ is montone and $F_n \to F$ pointwise.

By previous lemma,

$$F_n'(x) = f_n \quad \text{a.e.}$$

Therefore

$$\int_a^x F' \, dt \geq \int_a^x f_n \, dt = F_n(x) - F_n(a)$$

$$\int_a^x F' \, dt \geq \lim_{n \to \infty} F_n(x) - F_n(a)$$

$$= F(x) - F(a) = \int_a^x f \, dt = I(f).$$

$$\int_a^x F' \, dt \leq F(x) - F(a) = I(f) \quad \text{b/c } F \text{ is monotone.}$$

Hence $I(F') = I(f)$, and $F' = f$ a.e. by injectivity.

Proof (that D is injective). Suppose $F \in AC[a,b]$ and $F'(x) = 0$ a.e. then F is constant.

Let $\varepsilon > 0$, δ as in def of a.c., may assume $\delta < \varepsilon$.

let $\mathcal{B} = \{ U \subseteq [a,b] \text{ open st } \left| \frac{F(U)}{|U|} \right| < \frac{\delta}{n} \}$ for all n.

Since $F'(x) = 0$, and this is open, it is a covering, of the set where F' exists.

Then let $I_1 \dots I_N$ be intervals st. $m([a,b] - I) < \delta$.

Then $$|F(x) - F(a)| \leq \varepsilon + \sum_{I_n} |F(I_n)|$$

$$\leq \varepsilon + \sum \delta |I_n| \leq \varepsilon (1 + (b-a)) \to 0.$$ $\square$

12 iii) : Rectifiable Curves

Let $\gamma = (x(t), y(t))$ be such that

$$\|\gamma\|_{BV} < \infty.$$

Def 12.7: A curve satisfying the above is said to be <u>rectifiable</u>.

It's length is

$$L(\gamma) = \int_a^b |\dot{x}(t)^2 + \dot{y}(t)^2|^{1/2}\, dt < \infty.$$

Def 12.8: Let $K_\delta = \{(x,y) \mid \inf_t d((x,y), \gamma(t)) < \delta\}$.

The 1-dim Minkowski content of γ is $\lim_{\delta \to 0} \frac{m(K^\delta)}{2\delta}$.

Ex 12.9:
- The Koch snowflake is not rectifiable.
- A space filling curve has ∞ minkowski content.

Rem 12.10: Minkowski content leads to notion of "Hausdorff measure". This leads to "Geometric measure theory" and is key for studying singular limiting behavior of PDEs/minimal surfaces.

Thm 12.11 (Isoparametric Inequality)

Let $\Omega \subseteq \mathbb{R}^2$ be measurable, with $\partial\Omega$ rectifiable.

Then $$4\pi\, m(\Omega) \leq L(\gamma)^2,$$

with equality iff its a ~~disk~~ disk.

Lecture 13 Product measures and Fubini's Theorem

Let X be a metric space.

Def 13.1: An exterior measure on X is a function $\mu^*: 2^X \to [0,\infty]$ with

1) $\mu^*(\emptyset)=0$
2) $E_1 \subseteq E_2 \Rightarrow \mu^*(E_1) \leq \mu^*(E_2)$
3) $\mu^*(\bigcup_{j=1}^{\infty} E_j) \leq \sum_{j=1}^{\infty} \mu_*(E_j)$.

★ A set $E \subseteq X$ is said to be measurable if $\mu^*(A) = \mu^*(A \cap E) + \mu^*(A \cap E^c)$ for all $A \subseteq X$.

Prop 13.2: $\mathcal{M}(X) = \{E \subseteq X \mid E \text{ is measurable}\}$ forms a σ-algebra, and $\mu^* = \mu^*| : \mathcal{M}(X) \to \bar{\mathbb{R}}^{\geq 0}$ is a measure.

Proof: Nothing about previous proof used $\mathbb{R}$!

Def 13.3: A pre-measure on an algebra $A \subseteq 2^X$ (closed under finite $\cup, \cap$) is a function $\mu_0 : A \to [0,\infty]$ such that

1) $\mu_0(\emptyset) = 0$
2) $\mu_0(\bigsqcup_{k=1}^{\infty} E_k) = \sum_{k=1}^{\infty} \mu_0(E_k)$

the outer measure completing a premeasure is

$$\mu^*(E) = \inf \{ \sum_{j=1}^{\infty} \mu_0(E_j) \mid E \subseteq \bigcup E_j, E_j \in A \}.$$

Prop 13.4: μ^* as above is an exterior measure, and $A \subseteq \mathcal{M}(X)$ in the induced measure.

Thm 13.5: If $A \subseteq 2^X$ is a (covering) algebra (has countable covers of $\mu_0 < \infty$), and μ_0 a pre-measure, then $\exists!$ measure μ extending μ_0.

Proof: Existence immediate from 13.2 + 13.4. For uniqueness Suppose ν, μ are two measures extending μ_0.

If $F \subset \bigcup E_j$ w/ $E_j \in \mathcal{A}$,

$$\nu(F) \le \sum_{j=1}^{\infty} \nu(E_j) = \sum_{j=1}^{\infty} \mu_0(E_j)$$

$$\nu(F) \le \inf(\quad) = \mu(F).$$

and vice-versa. □

<u>Proof of 13.4</u>: μ is clearly an outer measure b/c

1) $\mu_*(\emptyset) = 0$, 2) $E_1 \subseteq E_2$ in A. means $\mu_0(E_1 \cup E_2 \setminus E_1) = \mu(E_1) + \underbrace{\mu_0(E_2)}_{\ge 0}$; so,

$$\inf\{\text{covers of } E_1\} \le \inf\{\text{covers of } E_2\}$$

b/c strictly bigger set.

3) $\mu^*\left(\bigcup_{j=1}^{\infty} E_j\right) = \mu^*\left(\bigsqcup\left(E_j \setminus \bigcup_{k=1}^{j-1} E_k\right)\right) = \sum \mu\left(E_j \setminus \bigcup_{k=1}^{j-1} E_k\right) \le \sum \mu(E_j)$.

On A, $\mu_*(E) \le \mu_0(E)$ since it covers itself

and if $E \subseteq \bigcup_{j=1}^{\infty} E_j$ then

$$\mu_0(E) = \sum\left(E_j \setminus \bigcup_{k=1}^{j-1} E_k\right) \le \sum \mu_0(E_k) \quad \text{and take infs.}$$

$A \subseteq \mathcal{M}(\mathbb{R})$ is similar. □

13.ii) Lebesgue Measure on $\mathbb{R}^n$

<u>Def 13.6</u>: Given (X, Σ, μ), $(Y, \mathcal{M}, \nu)$ the <u>product pre-measure</u> is the measure

$$\lambda(A \times B) = \mu(A)\nu(B)$$

on the algebra of products of measurable sets.

<u>Claim 13.7</u>: this is a pre-measure.

- $\lambda(\emptyset) = 0$ ✓
- If $A \times B = \bigsqcup A_j \times B_j$ all products of measurable, then (x, y) belongs to a single $A \times B_j$ for B_j depending on y. hence $B = \bigsqcup B_j$.

Then $\chi_A(x_1)\mu_2(B) = \sum_{j=1}^{\infty} \chi_{A_j}(x_1)\mu_2(B_j)$ for $x_1 \in A_j$.

Integrating

$$\mu(A)\mu(B) = \int \chi_A(x_1)\mu_2(B)\,dx_1 = \lim_{N\to\infty} \int \sum_{j=1}^{N} \chi_{A_j}(x)\mu_2(B_j)\,dx$$

$$= \lim_{N\to\infty} \sum_{j=1}^{N} \mu(A_j)\mu(B_j). \quad \square$$

<u>Def 13.8</u>: The product measure on $X \times Y$ is the one induced by the product pre-measure.

<u>Thm 13.9</u>: The product of Lebesgue measures on $\mathbb{R}^k \times \mathbb{R}^m$ is the Lebesgue measure on $\mathbb{R}^{m+k}$.

<u>Proof</u>: $k = m = 1$. Suffices to show that $\lambda = m^*$ product = Lebesgue outer on $A \times B$ measurable.

$A \subseteq \bigsqcup Q_j$, $B \subseteq \bigsqcup P_k$ cubes w/ difference ε.

$\Rightarrow$ $m(A \times B) \leq \sum_{Q_j, P_k} m(Q_j \times P_k) = \sum_{Q_j, P_k} m(Q_j)\, m(P_k)$ ← they agree on cubes!

$\leq \sum m(Q)m(P) \leq m(A)m(B) + \varepsilon$.

$\Leftarrow$ If Q_k (almost disjoint cubes) cover $A \times B$, then $Q_k = P_j \times M_i$. Therefore if $(x,y) \in A \times B$, $x \in P_j$, $y \in M_i$ some i so

$A \subseteq \bigcup P_j$ $B \subseteq \bigcup M_i$

$$m(A)m(B) \leq \sum_{j,i} m(P_j)\, m(M_i) = \sum m(Q_k) \leq m(A \times B) + \varepsilon. \quad \square$$

13.iii) Fubini's Theorem

<u>Thm 13.10</u> (Fubini 1) Suppose that $\int |f|\,dV < \infty$ on $\mathbb{R}^m \times \mathbb{R}^n$ where dV is the product Lebesgue measure. Then for a.e. $x \in \mathbb{R}^m$, $y \in \mathbb{R}^n$

- $f(x,-)$ is integrable on $\mathbb{R}^n$, $f(-,y)$ on $\mathbb{R}^m$; and

$$\int_{\mathbb{R}^{m+n}} f\,dV = \int_{\mathbb{R}^m}\left(\int_{\mathbb{R}^n} f(x,y)\,dy\right)dx = \int_{\mathbb{R}^n}\left(\int_{\mathbb{R}^m} f(x,y)\,dx\right)dy$$

Rem 13.11: Requires verifying $\int |f| dV < \infty$.

Thm 13.11 (Fubini II / Tonelli)
Suppose $|f| \geq 0$ on $\mathbb{R}^{m \times n}$ is measurable. Then $|f|(x,-)$, $|f|(-,y)$ are ~~integrable~~ measurable for a.e. x, y resp and

$$\int_{\mathbb{R}^{n+m}} |f| dV = \int_{\mathbb{R}^n} \left(\int_{\mathbb{R}^m} |f| \, dy \right) dx = \int_{\mathbb{R}^n} \left(\int_{\mathbb{R}^m} f(x,y) \, dx \right) dy$$

In particular, the hypotheses of Fubini's thm hold.

Proof: Stein-Shakarchi pg 76.

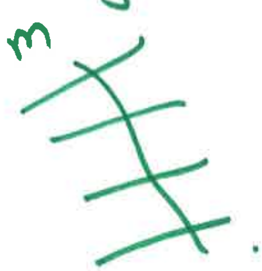
Corollary 13.12: If $E \subseteq \mathbb{R}^{m+n}$, then $E_x \cap \{x \times \mathbb{R}^n\}$ is measurable for almost every x, and

$$m(E) = \int_{\mathbb{R}^m} m(E_x) dx = \int_{\mathbb{R}^n} m(E_y) dy.$$

Proof: Tonelli on χ_E.

Corollary 13.13 (informal) If $M \subseteq \mathbb{R}^n$ is a submanifold, w/ codim $M \geq 1$, then M has measure 0.

Proof: After a change of variable, $\mathrm{Vol}(M) = \int_{\mathbb{R}^k \times \{0\} \subseteq \mathbb{R}^n} |\det \varphi| \, dx \, dy$.



$$= \int_{\mathbb{R}} \chi_M \, |\det \varphi| \, dx \, dy$$

$\downarrow$

$= 0$ for a.e. $y \in \mathbb{R}^{n-k}$.

Lecture 14 Radon-Nikodym Theorem

Let $(X, \mathcal{M}, \mu)$, $(X, \mathcal{M}, \nu)$ be two different measures on the same space.

Def 14.1: ν is said to be absolutely continuous with respect to μ if

$$E \in \mathcal{M}(X),\ \mu(E) = 0 \implies \nu(E) = 0.$$

Ex 14.2: Let $f \geq 0$ be continuous, bounded.

Set $$\nu(E) = \int_E f\,dx.$$

Clearly $m(E) = 0 \implies \nu(E) = 0$.
Same holds for f measurable (choose bounded approximation).

Lemma 14.3: ν is a.c. with respect to μ iff
$\forall \varepsilon > 0,\ \exists \delta$ such that

$$\mu(E) < \delta \implies \nu(E) < \varepsilon.$$

Proof: $\Rightarrow$ a.c. is obvious.

$\Leftarrow$ Suppose not. Then $\exists E_n$ w/ $\mu(E_n) \leq \frac{1}{2^n}$ but $\nu(E_n) \geq \varepsilon_0$.

Let $A_n = \bigcup_{k=n}^{\infty} E_n$, so $A_1 \supseteq A_2 \supseteq \dots$

Then $$\mu(A_n) \leq \frac{1}{2^{n-1}} \qquad \nu(A_n) \geq \varepsilon_0$$

Set $A = \bigcap_{n=1}^{\infty} A_n$. Then $\mu(A) \leq \mu(A_n)\ \forall n$
$= 0$.

But by cont. of measure $\nu(\bigcap A_n) = \lim \nu(A_n) \geq \varepsilon_0 \rightarrow \leftarrow$.

Def 14.4: Two measures are said to be mutually singular if
$\exists A, B$ disjoint s.t.

$$\nu(E) = \nu(A \cap E) \quad \mu(E) = \mu(B \cap E)$$

for all $E \in \mathcal{M}$

Fact 14.5. $L^2(\mathbb{R}, \mu)$ is a complete space in any measure.

Thm 14.6 (Radon-Nikodym)
Suppose μ is (σ-finite) and ν is absolutely continuous wrt μ. Then $\exists f \in L^1_{loc}(\mathbb{R}^n; \mu)$ such that

$$\nu(E) = \int_E f \, d\mu .$$

Rem 14.7: $f \in L^1_{loc}(X; \mu)$ means $f \in L^1(E)$ for any set of finite measure.

Ex 14.8: If $\nu = 2m$, then clearly $f = 2$ is not in $L^1(\mathbb{R}^n)$.

Def 14.9: On $\mathbb{R}$ $\nu(E) = \int_E g \, dx$ with $g \in \mathcal{C}^1$, then

Set $G(x) = \nu[0,x]$.

Then $G(x) = \int_0^x g \, dx$ so $G' = g$ by FTOC.

f is in general called the Radon-Nikodym derivative denoted $\frac{d\mu}{d\nu}$.

Proof: First assume X is a cube! Finite measures for both!
Let $\rho = \mu + \nu$. Let $\varphi \in L^2(X;\rho)^*$ be given by

$$\varphi(f) = \int_Q f \, d\nu.$$

Clearly $\|\varphi(f)\|_{\mathbb{R}} \leq \int_Q f \, d\rho \leq \|f\|_{L^2} \rho(X)$ so bounded.
By Reisz rep, $\exists g$ st.

$$\int_Q f \, d\nu = \int_Q f g \, d\rho.$$

__Claim__: $0 \leq g < 1$ a.e. wrt μ.
First, take $f = \chi_E$ for some E so,

$$\int_Q g \, d\rho = \int_Q \chi_E \, d\nu = \nu(E) < \mu(E) + \nu(E).$$

$<$ because if $\exists E$ w/ $\nu(E) > 0$, then equality implies $\mu(E) = 0$ → ← absolute cont. ☐

Now take $f = \chi_E(1 + g + \dots + g^n)$. Since

$$\int_Q f \, d\nu = \int_Q fg(d\nu + d\mu)$$

$$\int_Q f(1-g) \, d\nu = \int_Q fg \, d\mu$$

then
$$\int_Q f(1-g) \, d\nu = \int \chi_E \frac{(1-g^{n+1})}{1-g} 1-g \, d\nu = \int \chi_E (1-g^{n+1}) \, d\nu = \int g(1+g+\dots+g^n) \, d\mu$$

Now $1-g^{n+1} \to 1$ pointwise, D.C.T. $\Rightarrow$ $\nu(E) = \int \chi_E \, d\nu = \int f \, d\mu$, $f = \frac{g}{1-g}$.

$g + g^2 + \dots \to \frac{g}{1-g}$

For general case, cover

$$X = \bigcup E_j$$

with $\mu(E_j) < \infty$, $\nu(E_j) < \infty$ disjoint. Take $f = \sum f_j$. ◻.

<u>Thm 14.10</u> (Lebesgue Decomposition)

Let $(X, \mathcal{M}, \mu)$ e.g. $(\mathbb{R}^n, \mathcal{M}, m)$ be a σ-finite measure space. Let ν on $(X, \mathcal{M}, \nu)$ be another σ-finite measure. Then, $\exists!$ decomposition

$$\nu = \nu_a + \nu_s$$

s.t. ν_a is absolutely continuous wrt μ, ν_s is mutually sing.

<u>Proof</u>: In above proof, "atomic" part ν_s arises from $\{g = 1\}$. Take $B = \{x \in X \mid g(x) = 1\}$. Then

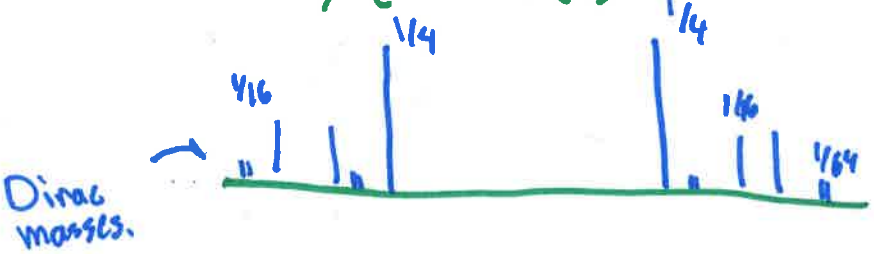
$$\nu_s(E) = \nu(E \cap B).$$

$$\mu(E) = \nu(E \cap B^c) \quad \text{mutually sing.}$$

<u>Ex 14.11</u>: Let μ_c be the measure

$$\mu_c([0,x]) = C(x) \quad \text{the cantor function}$$

Then $\mu_c =$ "$C'(x)$"



$x \in C$

$x = x_0 x_1 x_2 \ldots$

μ_c is mutually sing. w/ Lebesgue b/c $m(C) = 0$.

Lecture 15 | General Measures III: Interlude on probability

Def 15.1: A probability space $(X, \mathcal{F}, P)$ is a measure space such that $m(X) = P(X) = 1$. P is a probability measure.

Ex 15.2: Any discrete (finite) space S is a probability space where $\mathcal{F} = 2^S$, and $P(x_i) = p_i \geq 0$ s.t. $\sum p_i = 1$.

e.g. a 52-card deck w/ $p_i = \frac{1}{52}$.

Ex 15.3: For $X = \mathbb{R}$, the normal distribution

$$P = \frac{1}{\sqrt{2\pi}} e^{-x^2/2} \, dm \, x$$

is a probability space for $\mathcal{M} = \mathcal{F}$ the Lebesgue measurable sets.

$$P(E) = \int_E \frac{1}{\sqrt{2\pi}} e^{-x^2/2} \, dx.$$

More generally $X = \mathbb{R}^n$, $\mu \in \mathbb{R}^n$, Σ positive def $n \times n$

$$P = \frac{1}{(2\pi)^{n/2}} \frac{1}{\det \Sigma} e^{-\frac{1}{2}(x-\mu)^T \Sigma^{-1} (x-\mu)}.$$

15.i) Terminology Dictionary

Probability theory

Def 15.4: The following standard terminologies are equivalent

Measure theory (X, Σ, m)			Probability theory $(\Omega, \mathcal{F}, P)$	
$x \in X$	point	$\Rightarrow$	$x \sim P$, $x \in \Omega$	a sample
$E \subseteq X$	measurable	$\Rightarrow$	$E \subseteq \Omega$	an event
$m(E)$	measure	$\Rightarrow$	$P(E)$	probability of E.
$f: X \to \mathbb{R}$	measurable function	$\Rightarrow$	$X: \Omega \to \mathbb{R}$	random-variable

Ex 15.5: A random variable is an assignment of a number to each outcome. Eg $\Omega = \{P, N, D, Q\}$ coins,

$V : \Omega \to \mathbb{R}$ value

Ω = cards

$R : \Omega \to \mathbb{R}$ value ($J = 11$, $Q = 12$ ..)

Note χ_E is a R.V. $\forall E$ measurable.

Def 15.6: The expectation value or average of a random variable is

$$\mathbb{E}(f) = \int_\Omega f \, dP$$

e.g. $\mathbb{E}(f) = \sum_{x_i \in \Omega} f(x_i) p_i \qquad \mathbb{E}(f) = \int_{\mathbb{R}} f(x) \frac{e^{-x^2/2}}{\sqrt{2\pi}} dx$.

Lemma 15.7: If (X, Σ, μ) is a ~~probability~~ measure space, $Y \subseteq X$ measurable, then

$$(Y, \{Y \cap E \mid E \in \Sigma\}, \mu(E \cap Y))$$ is a measure space.

Proof: Obvious.

Def 15.8: If $(\Omega, \mathcal{F}, P)$ is a probability space, $E \in \mathcal{F}$ an event.

$\left(E, \mathcal{F} \cap E, \tilde{P} = \frac{P(- \cap E)}{P(E)}\right)$ is a probability space called the Conditional probability space.

$$P(A \mid E) = \tilde{P}(A) = \frac{P(A \cap E)}{P(E)}.$$

"the probability of A given E". the conditional probability

Corollary 15.9 (Bayès Rule) $P(E) P(A \mid E) = P(E \mid A) \cdot P(A)$.

Proof: Both are definitionally $P(A \cap E)$.

Rem 15.10: Radon-Nikodym theorem is used in defining conditional expectations.

15.ii) Some Inequalities

Thm 15.11: The following hold

1) Suppose $X : \Omega \to \mathbb{R}$ is a positive R.V.

$$P(X \geq a) \leq \frac{\mathbb{E}(X)}{a} \quad \text{(Markov Inequality)}$$

2) X any R.V.

$$P(|X - \mathbb{E}(X)| > a) \leq \frac{Var(X)}{a^2} \quad \text{(Chebyshev's Inequality)}$$

$Var = \mathbb{E}(|X - \mathbb{E}(X)|^2)$.

Proof: this is just translating notation.

15.iii) Large numbers and Central Limits

"The average of larger numbers of samples converges to the expectation value".

Def 15.12: Let $\Phi : (X, \Sigma, \mu) \to (Y, \Sigma', \_)$ a measurable mapping, the pushforward measure is defined by

$$\Phi_* \mu(B) = \mu(\Phi^{-1}(B)), \in \Sigma.$$

! Pushforward of R.V. is a measure containing Borel sets.

Def 15.13: Two random variables X_1, X_2 are said to be Identically distributed if

$$(X_1)_* P = (X_2)_* P. \quad \text{on } \mathbb{R}.$$

Def 15.14: Two random variables X, Y are said to be independent if

$$P(X \leq a) \cdot P(Y \leq b) = P(X \leq a \text{ and } Y \leq b) \text{ on } \Omega \times \Omega.$$

i.e. those measurable functions on $\Omega \times \Omega$ of the form $f(x)g(y)$.

Thm 15.15 (Law of Large Numbers) Suppose that $X_1, X_2 \ldots$ are a sequence of random variables that are identically distributed, and independent, w/

$$\mathbb{E}(X_i) = \mu \quad \forall i.$$

Then for $\overline{X} = \frac{1}{N}\sum_{i=1}^{N} X_i$

1) (Weak Law)

$$\overline{X}_N \longrightarrow \mu \quad \text{(the constant function)}$$

in probability (in measure)

2) (Strong Law)

$$\overline{X}_N \longrightarrow \mu$$

almost surely (pointwise a.e.).

Proof: Weak law assuming $Var(X_i) = \sigma^2 \ \forall i$.

$$Var(\overline{X}_N) = Var\left(\frac{1}{n}(X_1 + \ldots + X_N)\right)$$

ind.

$$= \frac{1}{n^2} Var(X_1 + \ldots + X_N) = \frac{1}{n^2}\sum Var(X_i) = \frac{\sigma^2}{n}$$

Then

$$P(|\overline{X}_N - \mu| \geq \varepsilon) \leq \frac{\sigma^2}{n\varepsilon^2} \quad \text{by Chebychev.}$$

$$\Rightarrow P(|\overline{X}_N - \mu| < \varepsilon) = 1 - \frac{\sigma^2}{n\varepsilon^2} \xrightarrow{n\to\infty} 0 \quad \forall \text{ fixed } \varepsilon. \quad \square$$

Thm 15.16 (Central Limit Theorem) Suppose X_i are independent, on $\mathbb{R}^n$ identically distributed, $\mathbb{E}[X_i] = \mu$ and $Var(X_i) = \sigma^2 < \infty$.

Then

$$\sqrt{N}(\overline{X}_N - \mu) \longrightarrow \frac{1}{\sqrt{2\pi}\sigma} e^{-x^2/2\sigma^2} = N(0, \sigma^2)$$

in distribution $\left(P(\ \leq a) \longrightarrow P(N \leq a) \ \forall a \in \mathbb{R}\right)$.

Rem 15.17: By the way, $P(Y \leq a) = \int_{-\infty}^{a} g \, dx$ where g is the R-N derivative wrt Lebesgue measure.

Lecture 16 Banach Spaces I: L^p, and the fundamental inequalities.

Ex 16.1: Consider the equivalence relation on E measurable,

$$f \sim g \text{ if } f = g \text{ a.e.} \iff \int_E |f-g|\,dx = 0.$$

Def 16.2: Let $(L^1(E), \| \ \|_1)$ denote the space of equivalence classes with the metric

$$d(f,g) = \|f-g\|_{L^1} := \int_E |f-g|\,dx$$

Lemma 16.3: $L^1(E)$ is a metric space.

Proof:
- $\|f-g\|_{L^1} = 0$ iff $f \sim g$ so $f = g$ in L^1 by definition.
- $\|f-g\|_{L^1} = \|g-f\|_{L^1}$ is clear.
- $\forall h$, by triangle inequality on $\mathbb{R}$,

$$\|f-g\|_{L^1} = \int_E |f-g| \leq \int_E |f-h| + |h-g| \leq \int_E |f-h| + \int_E |h-g|$$
$$\leq \|f-h\|_{L^1} + \|h-g\|_{L^1} \quad \square$$

Prop 16.4: $L^1(E)$ is complete, i.e., if $f_n \in L^1(E)$ is Cauchy, then $f_n \to f$ in $L^1(E)$ and $\|f_n - f\|_{L^1} \to 0$.

Proof: Note that if $|g_n| \geq 0$, and $\sum_{n=1}^{\infty} \int |g_n| < \infty$ then monotone convergence implies (apply to $G_N = \sum_{n=1}^{N} |g_n|$)

$$\sum_{n=1}^{\infty} \int |g_n| = \lim_{N \to \infty} \int G_n = \int \sum_{n=1}^{\infty} |g_n|.$$

In particular, $\sum_{n=1}^{\infty} |g_n|$ is integrable.

Now let f_{n_k} be a subsequence so $\|f_{n_k} - f_{n_{k+1}}\|_{L^1} \leq \frac{1}{2^k}$.

$$f = f_{n_1} + \sum_{k=1}^{\infty} f_{n_{k+1}} - f_{n_k}$$

$$g = |f_{n_1}| + \sum_{k=1}^{\infty} \underbrace{|f_{n_{k+1}} - f_{n_k}|}$$

By the above, $\int g < \infty$, since $\leq \frac{1}{2^k}$, so $f \in L^1$ integrable. Then claim.

$$f_{n_k} \to f \quad \text{a.e.}$$

$m\left(|f_{n_{k+1}} - f_{n_k}| > \frac{1}{\sqrt{2^k}}\right) \leq \frac{1}{\sqrt{2^k}} \int |f_{n_{k+1}} - f_{n_k}| \leq \frac{1}{\sqrt{2^k}}$ by Markov.

so $\sum m(|f_{n_{k+1}} - f_{n_k}|) < \infty \Rightarrow m(x \mid x \text{ in infinitely many}) = 0$

$m(x \mid f_{n_k} \text{ doesn't converge}) = 0$

by Borel - Cantelli.

Then $f_{n_k} \to f$ in L^1 by dominated convergence. Now choose N large

$$\|f_n - f\| \leq \|f_{n_k} - f_n\| + \|f_{n_k} - f\| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} \quad \square$$

16.ii) Banach Spaces

Def 16.5: A norm on a vector space is a function

$$\|\ \| : X \to \mathbb{R}^{\geq 0}$$

such that. $(X, \|\ \|)$ is a normed vector space.

i) $\|\alpha x\| = \alpha \|x\|$

ii) $\|x\| = 0$ iff $x = 0$

iii) $\|x+y\| \leq \|x\| + \|y\|$. Note $d_X(x,y) = \|x-y\|$ is therefore a metric.

Def 16.6: A normed vector space is a Banach Space if X is complete as a metric space.

Ex 16.7: $\ell^1 \subseteq \mathbb{R}^\infty$ is the space of sequences $a = \{(a_0, a_1, \dots)\}$

with $\|a\|_{\ell^1} = \sum_{i=1}^{\infty} |a_i|$

completeness follows from completeness of $\mathbb{R}$.

Ex 16.8 $\ell^2 = \{a_0, a_1 \quad \}$

$$\|a\|_{\ell^2} = \left(\sum_{i=1}^{\infty} |a_i|^2\right)^{1/2} \qquad \|a\|_{\ell^\infty} = \sup_i |a_i|.$$

$$\|a\|_{\ell^p} = \left(\sum_{i=1}^{\infty} |a_i|^p\right)^{1/p}$$

(all complete, different spaces)

Lemma 16.9: If $Y \subseteq X$ is a vector subspace closed in the induced topology then Y is also a Banach space. [16.3]

Ex 16.10: Let $K \subseteq \mathbb{R}$ be compact, then

$$\mathcal{C}^0(K) := \{ f \mid f \text{ continuous}, \ \|f-g\|_{\mathcal{C}^0} = \sup_x |f(x)-g(x)| \}$$

is a Banach space.

$$\mathcal{C}^{0,\alpha}(K) = \{ f \mid f \text{ Hölder continuous}, 0<\alpha\leq 1, \ \|f\|_{C^\alpha} = \sup |f(x)| + \sup_{x,y} \frac{|f(x)-f(y)|}{|x-y|^\alpha} \}$$

$\alpha = 1$ is Lipchitz continuous.

Def 16.11: Let E be a measurable set then the L^p spaces

$$L^p(E) = \{ f : E \to \mathbb{R} \mid f \text{ measurable}, \|f\|_{L^p} < \infty \} / \text{measure } 0$$

where $\|f\|_{L^p} = \left(\int_E |f|^p dx \right)^{1/p}$, is the L^p-norm

For $1 \leq p \leq \infty$, and

$$\|f\|_{L^\infty} = \inf \{ M \mid |f| \leq M \text{ a.e.} \}.$$

16.iii) Minkowski, Young, and Hölder Inequalities.

Thm 16.12 (Minkowski) If $1 \leq p < \infty$, $\|f+g\|_{L^p} \leq \|f\|_{L^p} + \|g\|_{L^p}$ w/ equality iff $f = \alpha g$.

Proof: It suffices to show the unit ball is convex, ie by linearity may assume $\|f\| + \|g\| = 1$. Then $\bar{f} = \frac{f}{\|f\|_{L^p}}$, $\bar{g} = \frac{g}{\|g\|_{L^p}}$

so $f + g = \alpha \bar{f} + \beta \bar{g}$. $\bar{f}, \bar{g} \in B$, and $\alpha + \beta = 1$,

so can show $\|f+g\| = \|\alpha \bar{f} + \beta \bar{g}\| \leq 1 = \|f\| + \|g\|$ ($\alpha \bar{f} + \beta \bar{g} \in B$).

Thus $\forall t \in [0,1]$ need $\|tf + (1-t)g\|_{L^p} \leq 1$. or $\|tf + (1-t)\|_{L^p}^p \leq 1$

$$\int |tf + (1-t)g|^p \leq \int t|f|^p + (1-t)|g|^p \leq 1 \quad (x^p \text{ is convex } \forall x.) \ \square$$

<u>Thm 16.13</u> (Reisz-Fisher) For $1 \le p \le \infty$, L^p is complete, and a Banach space.

<u>Proof</u>: The triangle inequality is precisely Minkowski's inequality.

For $1 < p < \infty$, the proof mimics $p=1$.

Suppose f_n is Cauchy, set $\|f_{n_k} - f_{n_{k+1}}\| \le \frac{1}{2^k}$

$$f = f_{n_1} + (f_{n_2} - f_{n_1}) + \ldots$$
$$g = |f_{n_1}| + | \qquad | + \ldots$$

then
$$\lim_{N \to \infty} \| G_N \|_{L^p} = \left(\int \left(\sum |g_n| \right)^p \right)^{1/p}$$
$$\le \sum_{n=1}^{N} \left(\int |g_n|^p \right)^{1/p}$$
$$< \infty.$$

(Minkowski obvious for $p=\infty$. (apply Minkowski N times))

So $g \in L^p \Rightarrow f \in L^p$. Rest same w/ Chebychev, and $|g|^p$ in dominated convergence. ∎

<u>Lemma 16.14</u>: If $m(A) < \infty$, and $f \in L^p, L^\infty$ for all p,
$$\lim_{p \to \infty} \|f\|_{L^p} = \|f\|_{L^\infty}$$

<u>Proof</u>: If χ is a step function
$$\|\chi_B\|_{L^p} = m(B)^{1/p} \to 1 = \|\chi_B\|_{L^\infty}$$

Now take limits. ∎

Lecture 17 | Banach Spaces II : Density, Young + Hölder | 17.1

Def 17.1 : A Banach space has a dense subset (subspace)

$$S \subseteq X$$

if $\forall x \in X, \varepsilon > 0 \ \exists s \in S$ with $\|s - x\| < \varepsilon$.

X is separable if $\exists$ a countable, dense subset.

Prop 17.2 : For $1 \le p \le \infty$ simple functions are dense.
For $1 \le p < \infty$ step, continuous, smooth functions are dense.

Proof : For L^∞, take f, so $m = \|f\|_{L^\infty}$. Divide $[-m, m]$ up and take $\sum_{n=1}^{N} (m - \frac{n}{N}) \cdot \chi_{f^{-1}[m - \frac{n}{N}, m - \frac{n+1}{N})}$

For $p \neq \infty$, let $f_M = \underset{\substack{(\max) \\ \text{if } <0}}{\min}[m, f] \cdot \chi_{[-m,m]}$.

Since $|f_m - f| \to 0$ pointwise, dominated by $|f|^p$

$$\int |f_m - f|^p \to 0.$$

If $S \subseteq T \subseteq X$ are dense inclusions, $S \subseteq X$ is dense.

For f_M use same trick as L^∞ (since $f_M \in L^1 \cap L^p \cap L^\infty$ and compact support).

smooth $\subseteq$ cont $\subseteq$ step $\subseteq$ simple. (Littlewood I.)

Prop 17.3 : L^p is separable for $p \neq \infty$.

Proof : Step functions $\sum_{i=1}^{N} q_i \chi_{[p_i, t_i]}$ $\quad q_i, p_i, t_i \in \mathbb{Q}$

are dense in step functions.

17.ii Young and Hölder Inequalities

Thm 17.4 : Suppose $\frac{1}{p} + \frac{1}{q} = 1$. Then

$$\|fg\|_{L^1(E)} \le \frac{\|f\|^p_{L^p(E)}}{p} + \frac{\|g\|^q_{L^q}}{q}.$$

Proof: It suffices to show

$$ab \le \frac{a^p}{p} + \frac{b^q}{q}.$$

(note $p=q=2$ is $2|ab| \le a^2+b^2$ is clear)

Since $\ln$ is concave, $t = \frac{1}{p}$ $1-t = \frac{1}{q}$

$$\ln(ta^p + (1-t)b^q) \ge \frac{1}{p}\ln(a^p) + \frac{1}{q}\ln(b^q)$$
$$= \ln(a) + \ln b = \ln(ab).$$

Take exp.

Corollary 17.5 (Young w/ ε) or "Peter-Paul" or "Absorption"

$\forall \varepsilon > 0$

$$\|fg\|_{L^1} \le \frac{\varepsilon^p \|f\|_{L^p}^p}{p} + \frac{\|f\|_{L^q}^q}{q\varepsilon^q}.$$

Proof: Apply Young to $fg = \varepsilon f \cdot \frac{g}{\varepsilon}$.

Rem 17.6: The above is, in my opinion, the most useful fact in analysis.

Thm 17.7 (Hölder's Inequality) Suppose $\frac{1}{p} + \frac{1}{q} = 1$. Then (including $p=1$, $q=\infty$)

$$\|fg\|_{L^1} \le \|f\|_{L^p}\|g\|_{L^q}$$

Proof: Set $\bar{f} = \frac{f}{\|f\|_{L^p}}$ $\bar{g} = \frac{g}{\|g\|_{L^q}}$.

Then $\|\bar{f}\bar{g}\|_{L^1} \le \frac{\|\bar{f}\|_{L^p}^p}{p} + \frac{\|\bar{g}\|_{L^q}^q}{q} = \frac{1}{p} + \frac{1}{q} = 1$. By Young.

$\Rightarrow$ $\|fg\| \le \|f\|_{L^p}\|g\|_{L^q}$

If $q=\infty$, $\int |fg| \le \sup f \cdot \int |g|$.

$\square$

17.iii) Examples

Ex 17.8: Suppose $m(E)$ is finite.

Then $q \geq p$,

$$\int |f|^p \leq \left(\int |f|^{p \cdot \frac{q}{p}} \right)^{p/q} \left(\int |1|^{q^*} \right)^{q^*}$$

$$\frac{1}{q^*} + \frac{1}{q/p} = 1.$$

$$\leq \|f\|_{L^q} \, m(E)^{q^*}$$

so finite L^q implies finite L^p so $L^q \subseteq L^p$.

Thus

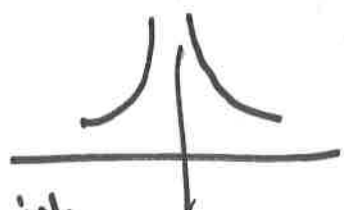
$$L^1(E) \supsetneq L^p(E) \supsetneq L^q(E) \supsetneq L^\infty(E)$$

are strict inclusions.

Intuition 17.9: Large p makes mass escaping vertically more problematic. Small p horizontally.

Ex 17.10: $E = \mathbb{R}$. $p \leq q$. Then $\exists f \in L^p$ $f \notin L^q$ and vice versa.

Take $f = \frac{1}{x^{1/q}} \chi_{[-1,1]}$.



$|f|^q = \frac{1}{x}$ not int, $|f|^p = \frac{1}{x^{p/q}}$ int.

Take $f = \frac{1}{x^{1/p}} \chi_{[1,\infty)}$

$|f|^q = \frac{1}{x^{q/p}}$ integrable

$|f|^p = \frac{1}{x}$ not integrable.

Rem 17.11: Recall every metric space has a completion $X \hookrightarrow \bar{X}$ = {equiv class of Cauchy Sequences}.

The density shows that

$$(\mathcal{C}^0[a,b], \| \ \|_{L^p}) \hookrightarrow L^p([a,b])$$

is this completion. So all we've done in some sense is very precisely characterize this completion.

17.iv) Dual Spaces

Def 17.12
A linear functional $\varphi : X \to \mathbb{R}$ is said to be bounded if

$$|\varphi(x)| \leq C\|x\| \quad \forall x \in X.$$

Def 17.13: Suppose X is a Banach space, then

$$X^* = \left(\{\varphi \mid \varphi : X \to \mathbb{R} \text{ bounded}\}, \|\varphi\|_{X^*} = \sup_{\|x\|=1} |\varphi(x)| \right)$$

Prop 17.14: X^* is a Banach space.

Proof: Linear functionals are linear, and triangle inequality on $\mathbb{R}$ shows norm.

Suppose φ_i is Cauchy, so that

$$\|\varphi_i - \varphi_j\|_{X^*} = \sup_{\|x\|=1} |\varphi_i(x) - \varphi_j(x)| \to 0.$$

Then $\varphi_i(x)$ is Cauchy $\forall x$, set $\varphi(x) = \lim_{i\to\infty} \varphi_i(x)$. It is a simple matter to show $\varphi(x)$ is linear. $\square$

Lemma 17.15: There is a natural Inclusion $X \hookrightarrow X^{**}$.
If it is an isomorphism, X is called Reflexive

Proof: Elements of $X^{**} = \{\xi : X^* \to \mathbb{R} \mid \text{bounded linear}\}$.

$$x \longmapsto \xi_x \quad \text{defined by } \xi_x(\varphi) := \varphi(x).$$

Then obviously linear,

$\|\xi_x(\varphi)\| \leq \|\varphi\| \|x\|$ so bounded. $\square$

Lecture 18 Banach Spaces III : Reisz Representation and Weak Topology

Recall each Banach space has a dual

$$X^* = \{ \varphi \mid \|\varphi(x)\|_X \leq \|\varphi\|_{X^*} \|x\| < \infty \}.$$

<u>Question 18.1</u> : What is the dual of $L^p(E)$.

<u>Ex 18.2</u> : There is an inclusion $L^q(E) \hookrightarrow (L^p(E))^*$ for $\frac{1}{p} + \frac{1}{q} = 1$.

$$f \longmapsto \underbrace{\int_E f \,\_\, dx}_{\varphi_f}.$$

It is bounded because by Hölder,

$$\|\varphi_f(g)\|_{\mathbb{R}} \leq \|f g\|_{L^1} \leq \|f\|_{L^q} \|g\|_p$$

So $\|\varphi_f\|_{(L^p)^*} = \|f\|_{L^q}$.

<u>Theorem 18.3</u> (Reisz Representation) If $\frac{1}{p} + \frac{1}{q} = 1$ w/ $1 < p, q < \infty$,

then

$$L^q(E) \simeq L^p(E)^*$$
$$f \longmapsto \varphi_f$$

is an isometry.

! We will prove for $E = \mathbb{R}$ or $[a,b)$

<u>Corollary 18.4</u> : L^p is reflexsive.

<u>Proof</u> : Given $\varphi \in L^p(E)^*$ define a candidate

$$F(x) = \begin{cases} \varphi(\chi_{[0,x]}) & x > 0 \\ -\varphi(\chi_{[x,0]}) & x < 0 \end{cases}$$

<u>Claim</u> F is absolutely cont.

$$\sum_{i=1}^N |F(b_i) - F(a_i)| = \left| \sum_{i=1}^N \varphi(\chi_{[a_i,b_i]}) \right|$$

$$\leq \varphi\left(\sum_{i=1}^n \chi_{[a_i,b_i]}\right) \leq C_\varphi \left\| \sum_{i=1}^n \chi_{[a_i,b_i]} \right\|_{L^p}$$

$$\leq C_\varphi \delta^{1/p}.$$

Thus $\forall\ [n,n+1]\ \exists f$ st $f(x) = F'(x)$ a.e.

Then

$$\varphi_f(\chi_{[a,b]}) = \int f\chi_{[a,b]}\,dx = \int_a^b f\,dx = F(b)-F(a) = \varphi(\chi_{[a,b]})$$

agree on step functions + continuity / density.

It remains to show $f \in L^q(\mathbb{R})$. Suppose first $f \geq 0$.

Let $f_M = \min(M,f)\,\chi_{[-M,M]}$.

$$\int_{\mathbb{R}} |f_M|^q \leq \int_E |f_M|^{q-1}|f_M| \leq \varphi_f(|f_M|^{q-1})$$

$$\leq C_\varphi \| f_M^{q-1} \|_{L^p}$$

$$\leq C_\varphi \left(\int |f_M|^{(q-1)p}\right)^{1/p}$$

$$\leq C_\varphi \left(\int |f_M|^q\right)^{1/p}$$

$\frac{1}{p} + \frac{1}{q} = 1 \Rightarrow p+q = pq$

so $pq - p = q$

so $\| f_M \|_{L^q}^{q(1-\frac{1}{p})} \leq C_\varphi$. ($q(1-\frac{1}{p}) = q \cdot \frac{1}{q} = 1$.)

Then $M \to \infty$ by monotone convergence, $\|f\|_{L^q} \leq C_\varphi$.

For f general use $f = f^+\chi^+ - f^-\chi^-$ trick. □

__Thm 18.5__ (Reisz Rep $p = \infty$) $L^1(\mathbb{R})^* = L^\infty(\mathbb{R})$, but not vice versa. ↑ extend δ_0.

__Proof__: $F(x)$ as before,

$$\|F(b)-F(a)\|_{\mathbb{R}} \leq \|\varphi\|_* \|\chi_{[b,a]}\|_{L^1} = (b-a)\|\varphi\|_*$$

so absolutely cont as before,

$$f = F' \quad \text{a.e.}$$

and $\|f\|_{L^\infty} \leq \|\varphi\|_*$.

Again $\varphi_f(\chi_{[a,b]}) = \int_a^b F' = F(b)-F(a) = \varphi(\chi_{[a,b]})$

agree, so equal by density. □

18.ii) The Weak Topologies

Def 18.6 a sequence $f_n \in X$ to <u>converge weakly</u> to f
$$f_n \rightharpoonup f$$
if $\varphi(f_n) \to \varphi(f)$ in $\mathbb{R}$ $\forall \varphi \in X^*$.

Ex 18.7: $f_n = \chi_{[n,n+1]} \rightharpoonup 0$ in $L^2(\mathbb{R})$ but not strongly. By Reisz $(L^2)^* \simeq L^2$

So $$\int_{\mathbb{R}} \chi_{[n,n+1]}\, g \leq \|g\|_{L^2[n,n+1]}$$
but $\sum_{n=1}^{\infty} \|g\|_{L^2([n,n+1])} < \infty$ so $\to 0$.
but $\|f_n\|_{L^2} = 1$ $\forall n$.

Prop 18.8: If $f_n \rightharpoonup f$ in $L^p(E)$ $1 \leq p < \infty$ then $\exists M$ st $\|f_n\|_{L^p} < M$.

Proof: Suppose not, By passing to a subsequence, May assume $f=0$. $\|f_n\| \geq n \cdot 3^n$.

Renormalize so $$g_n = \frac{n \cdot 3^n}{\|f_n\|} \cdot f$$ (bounded by $[0,1]$ so sub converges.)

Claim $\|g_n\| = n \cdot 3^n$ and $g_n \to g$ weakly $\varphi(g_n) = \frac{n \cdot 3^n}{\|f_n\|}\varphi(f_n)$

Now let $\varepsilon_1 = \frac{1}{3}$
$$\varepsilon_{n+1} = \frac{1}{3^{n+1}} \cdot \operatorname{sign}\left(\int_E \left[\sum_{k=1}^{n} \varepsilon_k (f_n)^*\right] f_n\right)$$

$f_n^* = |f_n|_p^{p-1} \operatorname{sgn}(p) |f|^{p-1} \geq 0$, $\in L^q$, $\left(\int f_n^* f_n\right)^{1/p} = \|f_n\|_p$

so $$\left|\int_E \sum_{k=1}^{N} \varepsilon_k f_k^* f_n\right| \geq \frac{1}{3^n}\|f_n\|_p = n$$
$\|f_n^*\|_q = 1$ so $\|\varepsilon_n f_n^*\|_q = \frac{1}{3^n}$

Thus $\sum_{k=1}^{\infty} \varepsilon_k f_k^* \longrightarrow g$ in L^q.

$$\left|\int g \cdot f_n\right| \geq \left|\int \sum_{k=1}^{\infty} \varepsilon_k f_k^* f_n\right| - \int \sum_{k=n+1}^{\infty} \varepsilon_k f_k^*$$

$$\geq n - \left\| \sum_{k=n+1}^{\infty} \varepsilon_k f_k^* \right\|_q \|f_n\|_p$$

$$\geq n - \frac{1}{3^{n-1}} \|f_n\|_p \quad \not\to 0. \quad \square$$

Thm 18.9 (Banach-Alaoglu): Let $1 < p < \infty$. Then a sequence $\{f_n\}$ has a weakly convergent subseq iff it has a bounded subsequence.

ie "The unit ball is weakly compact".

Proof: See Royden 8.3.

Rem 18.10: This is a great result. If you want to show

$f_k \to f$ (say the minimum of a functional or sol of PDE)

Then can split as two easier results

1) f_k is bounded $\Rightarrow \exists f$ st $f_k \rightharpoonup f$.
2) convergence is actually strong (no escaping mass).

Lecture 19 Fourier Series I: Hilbert spaces.

Def 19.1: A linear operator $L: X \to Y$ on Banach spaces is said to be bounded if $\exists C = C_L$ such that

$$\|Lx\|_Y \leq C_L \|x\|_X.$$

Lemma 19.2: A linear operator is continuous iff it is bounded.

Proof: $\forall \varepsilon > 0$ take $\delta = \frac{\varepsilon}{C_L}$ so

$$\|Lx - Ly\|_Y = \|L(x-y)\|_Y \leq C_L \|x-y\| \leq \varepsilon \quad \text{if } \|x-y\| < \delta.$$

Take $\varepsilon = 1$. $\exists \delta$ s.t. if $\|x - 0\| < \delta$ then for $\|x\|_X = \delta$

$$\|Lx\|_Y \leq 1 \leq \frac{1}{\delta}\|x\|_X. \quad \text{and both sides scale.}$$

Def 19.2: Two Banach spaces X_1 & X_2 are isomorphic if $\exists$ a bounded linear isomorphism w/ bounded inverse.

19.i) Hilbert Spaces

Def 19.3: An inner product space H with

$$\langle \, , \rangle : H \times H \to \mathbb{R}$$

bilinear is a Hilbert space if the norm $\|h\|_H = \langle h, h \rangle$ makes it into a Banach space.

Ex 19.4: $L^2(\mathbb{R})$ is a Hilbert space with

$$\langle f, g \rangle_{L^2} = \int_{\mathbb{R}} fg \, dx$$

Hölder's inequality specializes to Cauchy-Schwartz

$$\S \quad \langle f, g \rangle_{L^2} \leq \|f\|_{L^2} \|g\|_{L^2}$$

Rem 19.5: In the case of a general inner product space Cauchy-Schwartz follows from general nonsense.

Def 19.6: A collection of vectors $\{e_i\}$ is said to be an orthonormal set if $\langle e_i, e_j \rangle = \delta_{ij}$.

Def 19.7: An orthonormal set is said to be a basis if it is complete, i.e. the equivalent conditions hold

1) If $\langle x, e_i \rangle = 0 \ \forall i$, then $x = 0$.
2) $\sum_{i=1}^{N} a_i e_i$ finite linear combinations are dense.

Proof: Suppose dense. $\|x - \sum_{i=1}^{N} a_i e_i\| < \varepsilon$, but $0 = \langle x, \sum a_i e_i \rangle$
$= \langle x, (\sum a_i e_i - x) + x \rangle$
$= \|x\| - \varepsilon \|x\| \rightarrow \Leftarrow$

Suppose complete, if not dense $\exists\, y$ w/
$$\| y - \sum_{i=1}^{N} a_i e_i \| > \delta \quad \forall a_i, N \text{ for some } \delta.$$
Then $y - \sum_{i=1}^{\infty} \langle y, e_i \rangle$ has $\langle y, e_i \rangle = 0 \ \forall i \rightarrow \Leftarrow$.

Algorithm 19.8 (Gram-Schmidt)

Let a_i be a dense subset of H, for $i = 1, 2, \ldots$

Set
$$e_1 = a_1$$
$$\tilde{e}_{n+1} = a_{n+1} - \sum_{i=1}^{n} \langle a_{n+1}, e_i \rangle \quad \text{then } e_i := \frac{\tilde{e}_i}{\|\tilde{e}_i\|}.$$

Corollary 19.9: A separable Hilbert space has a (countable) orthonormal basis, hence
$$x = \sum_{i=1}^{\infty} \langle x, e_i \rangle e_i \quad \forall x \in H.$$

Proof: Existence follows from Gram-Schmidt on dense subspace.

Note $a_i = \langle x, e_i \rangle$ has $\sum |a_i|^2 \leq \|x\|$

Then $0 \leq \|x - \sum a_i e_i\| = \|x\|^2 + \sum |a_i|^2 - 2\langle x, \langle x, e_i \rangle e_i \rangle$
$= \|x\|^2 - \sum |a_i|^2$

So $\sum_{i=1}^{\infty} \langle x, e_i \rangle e_i$ converges. By completeness, $x -$ this $= 0$.

Corollary 19.10: $\exists$ an isomorphism $H \longrightarrow \ell^2(\mathbb{N})$ for each basis (in fact an isometry)

Proof: $x \longrightarrow \{\langle x, e_i \rangle\}_{i=1}^{\infty}$

Thm 19.11 (Abstract Reisz Representation) Let H be a separable Hilbert space, then
$$H \to H^*$$
$$x \mapsto \langle x, - \rangle$$
is an isomorphism.

Proof: Obviously linear, and $\sup_{\|f\|=1} \langle x,f \rangle = \|x\|$ so bounded, and obviously inj.

Let $\varphi \in H^*$. Set $a_i := \varphi(e_i)$. Then
$$(\varphi - \langle \Sigma a_i e_i \,) e_j = \varphi(e_j) - a_j = 0$$
so $\varphi = \sum_{i=1}^{\infty} a_i e_i$, hence surj. (isometry so inverse is bounded).

19.ii) Examples

Ex 19.12: $\ell^2(\mathbb{N})$ or $\ell^2(\mathbb{Z})$ with bases $e_i = \{0,0,0 \ldots 1, 0, \ldots\}$

Ex 19.13: The quantum mechanics Hilbert space is (usually) $L^2(\mathbb{R}^3)$.

Ex 19.14: More generally (X, Σ, μ) any measure space $L^2(X;\mu)$.

Ex 19.15 $L^2[-1,1]$ a basis is the Legendre polynomials
$1, x, \frac{1}{2}(3x^2-1), \frac{1}{2}(5x^3-3x), \frac{1}{8}(35x^4-30x^2+3) \ldots$ $= \frac{1}{2^n n!}\left(\frac{d}{dx}\right)^n (x^2-1)^n$
(Gram-Schmidt on x^n), density below.

Ex 19.16: $L^2(\mathbb{R}, e^{-x^2/2})$
$$h_n(x) = (-1)^n e^{x^2}\left(\frac{d}{dx}\right)^n e^{-x^2} = \left(2x - \frac{d}{dx}\right)^n \cdot 1.$$
$$= 1,\ 2x,\ 4x^2-2,\ 8x^3-12x, \ldots$$
are orthonormal basis.

(Hermite Polynomials)

19.iii) Fourier Series

Def 19.17: an algebra of functions on K compact
1) separates points if $\forall x,y \in K$, $\exists f$ w/ $f(x) \neq f(y)$
2) vanishes nowhere if $\nexists x \in K$ w/ $f(x) = 0$ $\forall f$.

Thm 19.18 (Stone-Weierstrass) Any algebra $\mathcal{F} \subseteq \mathcal{C}^0(K)$ that separates points and vanishes nowhere is dense in $\mathcal{C}^0(K)$.

Upgrade 19.19: Consider $\mathbb{C}$-valued functions.

$L^2([a,b])$ works same w/ $\langle f,g \rangle = \int_a^b f\bar{g}\, dx$

Also $L^2([a,b]) \supseteq L^2(S^1)$ as completion of $f(a) = f(b)$ in $\mathcal{C}^0$.

➡ For next few lectures we work with $L^2(S^1; \mathbb{C})$, $\langle f,g \rangle = \frac{1}{2\pi}\int_0^{2\pi} f\bar{g}\, d\theta$.

Thm 19.20: $\{e^{in\theta}\}_{n=-\infty}^{\infty}$ for $\theta \in [0, 2\pi]$ is an orthonormal basis for $L^2(S^1; \mathbb{C})$.

Proof:
- $\frac{1}{2\pi}\int_0^{2\pi} |e^{in\theta}|^2 d\theta = \frac{1}{2\pi} \cdot 2\pi = 1$
- $\frac{1}{2\pi}\int_0^{2\pi} \langle e^{in\theta}, e^{im\theta} \rangle d\theta = \frac{1}{2\pi}\int_0^{2\pi} e^{i(n-m)\theta} = \frac{1}{i(n-m)} e^{i(n-m)\theta}\Big]_0^{2\pi} = 0$.
- $\mathrm{Re}(e^{in\theta})$ and $\mathrm{Im}(e^{in\theta})$ never simultaneously vanish
- ~~Since const $\in \mathcal{C}^0$, if $\exists x \neq y$ then $\exists x, y$ st $f(x) = f(y) = 0$~~

  $e^{i\theta}: S^1 \to S^1 \subseteq \mathbb{C}$ is already injective.

Def 19.21: The isomorphism / isometry

$$\mathcal{F}: L^2(S^1; \mathbb{C}) \longmapsto \ell^2(\mathbb{Z}; \mathbb{C})$$
$$f \longmapsto \langle e^{in\theta}, f \rangle$$

is called the Fourier Series, and denoted

$$\mathcal{F}(f) = \hat{f}(n) : \mathbb{Z} \to \mathbb{C}.$$

Lecture 20 Fourier Series II : Derivatives and convolutions

Recall Fourier series gave an isomorphism

$$L^2(S^1;\mathbb{C}) \overset{\mathcal{F}}{\cong} \ell^2(\mathbb{Z};\mathbb{C})$$
$$f \longmapsto \hat{f}(n) = \{\langle f, e^{in\theta}\rangle_{L^2}\}$$
$$\sum_{n=-\infty}^{\infty} c_n e^{in\theta} \longleftarrow \{c_n\}$$

In fact

__Thm 20.1__ (Plancharel) $\mathcal{F}$ is an isometry,

$$\|f\|_{L^2} = \sum_{n\in\mathbb{Z}} |c_n|^2 \qquad \hat{f}(n)=c_n.$$

__Ex 20.2__ : (Real Fourier Series) For $c_n = a_n + ib_n$

$$c_n e^{in\theta} = (a_n+ib_n)(\cos(n\theta)+i\sin(n\theta))$$
$$= a_n\cos(n\theta) - b_n\sin(n\theta) + i\,(b_n\cos(n\theta)+a_n\sin(n\theta)).$$

And

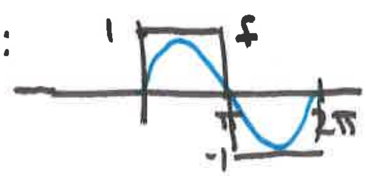
$$\langle f+ig, \cos(n\theta) \mp i\sin(n\theta)\rangle_{\mathbb{C}} = \langle f,\cos\rangle + \langle g,\sin\rangle + i\langle f,-\sin\rangle + i\langle g,\cos\rangle.$$

$\Rightarrow$ if f is real,

$$f = \mathrm{Re}\cdot\Big(\sum_n c_n e^{in\theta}\Big) = \sum_n a_n\cos(n\theta) + -b_n\sin(n\theta)$$

where
$$a_n = \langle f, \cos(n\theta)\rangle_{L^2}$$
$$b_n = \langle f, -\sin(n\theta)\rangle_{L^2}$$

__Ex 20.3__ :



then $a_n = 0 \quad \forall n$ b/c function is odd.

$$-b_n = \frac{1}{2\pi}\int_0^{2\pi} f(x)\sin(nx)\,dx$$
$$= \frac{1}{\pi}\int_0^{\pi} \sin(nx)\,dx$$
$$= \frac{1}{\pi n}\Big[-\cos(nx)\Big]_0^{\pi} = \begin{cases} \frac{2}{\pi n} & n \text{ even} \\ 0 & n \text{ odd}\end{cases}$$

$$f = \sum_{n \text{ odd}} (-1)^n \frac{2}{\pi n}\sin(nx).$$
$$= \sum_{n=1,3,5} \frac{4}{\pi n}\sin(nx).$$

20.i) The Derivative

Observe $-i\partial_\theta e^{in\theta} = n e^{in\theta}$.

Prop 20.4: The Fourier series satisfies (for $f \in C^\infty$)

$$\mathcal{F}(-i\partial_\theta f) = n\hat{f}(n)$$

Proof: Let $g = -i\partial_\theta f \in L^2$.

$$g = \sum_{n=-\infty}^{\infty} d_n e^{in\theta} \quad \text{where}$$

$$f = \sum_{n=-\infty}^{\infty} c_n e^{in\theta} \quad \text{where} \quad c_n = \frac{1}{2\pi}\int_0^{2\pi} \langle f, e^{in\theta}\rangle d\theta$$

And
$$d_n = \frac{1}{2\pi}\int_0^{2\pi}\langle -i\partial_\theta f, e^{in\theta}\rangle d\theta = \frac{1}{2\pi}\int_0^{2\pi}\langle \partial_\theta f, ie^{in\theta}\rangle d\theta$$
$$= \frac{1}{2\pi}\int_0^{2\pi}\langle f, -i\partial_\theta e^{in\theta}\rangle d\theta$$
$$= n \cdot c_n$$

Hence $\sum_{n=-\infty}^{\infty} |n|^2 |c_n|^2 < \infty$, and $\left\| \sum_{n=-N}^{N} n c_n e^{in\theta} - (-i\partial_\theta f)\right\|_{L^2} \xrightarrow{N\to\infty} 0$.

Corollary 20.5: $\left(-i\frac{d}{d\theta}\right)^k f \in L^2$ if and only if $\sum |n|^{2k}|c_n|^2 < \infty$.

Proof: $\Rightarrow$ induction on above.

$\Leftarrow$ $\lim_{h\to 0}\int \left|\frac{f_N(\theta+h)-f(\theta)}{h}\right|^2 d\theta$

$-i\partial_\theta f_N \to g$ in L^2. Claim $g = -i\partial_\theta f$. exchange limits + dominated convergence.

20.ii) The Convolution

Question 20.6: What is $\mathcal{F}(fg)$?

Thm 20.7: If $f, g \in L^2$ with ~~...~~ ~~then~~ $g \in C^1(S^1; \mathbb{C})$. Then

$$\mathcal{F}(fg) = \hat{f} * \hat{g}$$
$$= \sum_{n\in\mathbb{Z}}\sum_{k\in\mathbb{Z}} \hat{f}(n-k)\hat{g}(k)$$

Proof : First note

$(\hat{f} * \hat{g}) \in \ell^2$, as

$$\|\hat{f} * \hat{g}\|_{\ell^2} = \sum_n \left| \sum_{k \in \mathbb{Z}} \hat{f}(n-k)\hat{g}(k) \right|^2$$

$$\leq \sum_n \left(\left(\sum_k \left| \frac{\hat{f}(n-k)}{(k^2+1)} \right|^2 \right)^{1/2} \left(\sum_k (k^2+1)|\hat{g}(k)|^2 \right)^{1/2} \right)^2$$

$$\leq \sum_n \left(\|g\|_{L^2} + \|dg\|_{L^2} \right) \sum_k \frac{|\hat{f}(n-k)|^2}{|k^2+1|^2}$$

$$\leq \sum_k \sum_n \frac{|\hat{f}(n-k)|^2}{(k^2+1)^2} \leq \left(\|g\|_{L^2} + \|dg\|_{L^2} \right)\left(\|f\|_{L^2} \right) \sum \frac{1}{k^n}$$

$< \infty$.

And

$$\langle fg, e^{in\theta} \rangle = \langle f \cdot \sum_{k \leq N} \hat{g}_k e^{ik\theta}, e^{in\theta} \rangle + \langle f \cdot \sum_{k \geq n} \hat{g}_k e^{ik\theta}, e^{in\theta} \rangle$$

$$= \langle f, \sum e^{i(n-k)\theta} \hat{g}_k \rangle +$$

$$= \sum_{k \leq N} \hat{g}_k \hat{f}_{n-k} + \qquad \leq \|f\|_{L^2} \|\sum_{k \geq N} \hat{g}_k\|$$

$$\leq \|f\|_{L^2} \cdot \frac{1}{N^2}\left(\|g\|_{L^2} + \|dg\|_{L^2} \right)$$

$$\Rightarrow \langle fg, e^{in\theta} \rangle = \sum_{k \in \mathbb{Z}} \hat{g}(k)\hat{f}(n-k).$$

□

Thm 20.8 : $\mathcal{F}(f * g) = \hat{f} \cdot \hat{g} \cdot 2\pi$ when all are defined.

Proof :

$$f * g = \int_0^{2\pi} f(\theta - \tau) g(\tau)\, d\tau$$

$$\widehat{f * g}_n = \frac{1}{2\pi}\int_0^{2\pi}\int_0^{2\pi} f(\theta-\tau) g(\tau) e^{-in(\theta + \tau - \tau)}\, d\tau\, d\theta$$

$$= \frac{1}{2\pi}\int_0^{2\pi}\int_0^{2\pi} f(\theta-\tau) g(\tau) e^{-in\tau} e^{-in(\theta-\tau)}\, d\tau\, d\theta$$

$\theta' = \theta - \tau$

$$= \frac{1}{2\pi}\int_0^{2\pi}\int_0^{2\pi} g(\tau) e^{-in\tau} f(\theta') e^{-in\theta'}\, d\theta'\, d\tau \qquad \text{Fubini}$$

$$= \int_0^{2\pi} \hat{g}(n) f(\theta') e^{-in\theta'}\, d\theta' = 2\pi \hat{f}(n)\hat{g}(n).$$

20(iii) Approximate Identities

Let $\delta \in C^0(S^1)^*$ be s.t. $\delta(f) = f(0)$. Written

$$\delta(f) := \frac{1}{2\pi}\int_0^{2\pi} \delta(x) f(x)\, dx.$$

Then

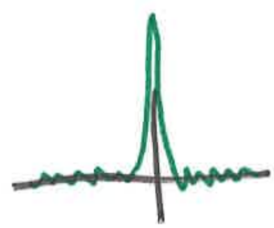
$$f(x) = f * \delta(x) = \frac{1}{2\pi}\int_0^{2\pi} \delta(y) f(x-y)\, dy = f(x).$$

Intuitively,

$$\mathcal{F}(\delta) = \frac{1}{2\pi}\int_0^{2\pi} \delta(x) e^{inx} dx = \frac{1}{2\pi} \ \forall n, \text{ and } \|\delta(x)\|_{L^2} = \infty \notin L^2$$

Def 20.9: The Dirichlet Kernel for $N \in \mathbb{N}$

$$D_N(x) = \frac{1}{2\pi}\sum_{-N}^{N} e^{inx}$$



Thm 20.10 (Approx Identity): 1) $\lim_{N\to\infty} f * D_N = f$ in L^2 $\forall f \in L^2$

2) $f * D_N \in C^\infty$ and $\|\partial^m (f*D_N)\|_{L^2} \leq N^m \|f\|_{L^2}$

Proof: 1) $\widehat{f*D_N} = \chi_{[-N,N]}\hat{f}$ so $\|f*D_N - f\|_{L^2} = \|\sum_{n\geq N} c_n e^{in\theta}\| \to 0$.

2) $\|\partial^m (f*D_N)\|_{L^2}^2 = |\sum |n|^{2m} \chi_{[-N,N]} \hat{f}|^2$

$$\leq N^{2m} |\sum_n |\hat{f}_n|^2 \leq N^{2m} \|f\|_{L^2}.$$

and $\partial^m f \in L^2$ $\forall m \Rightarrow f$ is smooth.

Lecture 21: Convergence of Fourier Series (Fourier Series III)

Question 21.1: If $f \in L^2$ then $f_N = \sum_{|n| \le N} \hat{f}(n) e^{in\theta} = f * D_N$ converges to f in L^2.

Under what conditions does $f_N \to f$ pointwise? uniformly?

Ex 21.2 (Fejér): There exists an $f \in C^0(S^1)$ such that $f_N(0) \to \infty$.

$$f(x) = \sum_{k=1}^{\infty} \sin\left((2^{k^3}+1)\frac{x}{2}\right) \cdot \frac{1}{k^2}$$

Then $f(x)$ is a sup norm-limit of C^0, so C^0, but

$$\lim_{N\to\infty} f_N(0) = \sum_{n=1}^{\infty} a_n$$

$$= \sum_n \frac{2}{2\pi} \sum_{k=1}^{\infty} \frac{1}{k^2} \int_0^{\pi} \sin\left((2^{k^3}+1)\frac{\theta}{2}\right) \cos(n\theta)\, d\theta$$

$$= \frac{1}{\pi} \sum_{n=1}^{N} \sum_{k=1}^{\infty} \frac{1}{k^2} \cdot \lambda_{n, \frac{1}{2}k^3+1} = \frac{1}{\pi} \sum_k \frac{\sigma_k}{k^2}$$

Can show $\sigma_k = \sum_{n,k=1}^{N} \lambda_{nk}$ has $\sigma_{kN} = \sum_{n=1}^{N} \lambda_{nN} \ge \frac{1}{2} \log(N)$

$\Rightarrow$ For $N \ge 2^{k^3} - 1$, $\quad f_N(0) \ge \frac{1}{k^2} \cdot \frac{1}{2} \log(2^{k^3}+1) \ge \frac{k}{2}$.

21.i) Sobolev Spaces

Since $\widehat{-i\partial_\theta f} = n\hat{f}(n)$ we see

$$\partial_\theta^k f \in L^2 \iff n^k \hat{f} \in \ell^2$$

Define 21.3 the (continuous) resp (discrete) Sobolev norm of regularity k by

$$\|f\|_{L^{k,2}} = \left(\int_0^{2\pi} |f|^2 + |\partial f|^2 + |\partial^2 f|^2 + \dots |\partial^k f|^2 \, d\theta \right)^{1/2}$$

$$\|\hat{f}\|_{\ell^{k,2}} = \left(\sum_{n=-\infty}^{\infty} (1 + |n|^2 + |n|^4 + \dots + |n|^{2k}) |\hat{f}(n)|^2 \right)^{1/2}.$$

Def 21.4
Define the Sobolev spaces

$$L^{k,2}(S^1; \mathbb{C}) \quad , \quad \ell^{k,2}(\mathbb{Z}; \mathbb{C})$$

are the completions of C^∞, finite seq in the above norms.

Thus $f \in L^{k,2}$ iff it has k-derivatives in L^2 * this needs "weak derivatives" to be made precise

Corollary 21.5: The following diagram commutes

$$\begin{array}{ccc}
L^2(S^1;\mathbb{C}) & \overset{\mathcal{F}}{\longleftrightarrow} & \ell^2(\mathbb{Z};\mathbb{C}) \\
\cup\!\!\uparrow & & \cup\!\!\uparrow \\
L^{1,2}(S^1;\mathbb{C}) & \overset{\mathcal{F}}{\longleftrightarrow} & \ell^{1,2}(\mathbb{Z};\mathbb{C}) \\
\cup\!\!\uparrow & & \cup\!\!\uparrow \\
L^{2,2}(S^1;\mathbb{C}) & \overset{\mathcal{F}}{\longleftrightarrow} & \ell^{2,2}(\mathbb{Z};\mathbb{C}) \\
\uparrow & & \cup\!\!\uparrow \\
\vdots & &
\end{array}$$

and each horizontal arrow is an isometry in the $L^{k,2}$, $\ell^{k,2}$ norms.

Proof: inclusions commute w/ $\mathcal{F}$ so commuting is obvious.

As before

$$\int |f|^2 + |\partial f|^2 \, d\theta = \sum_{n\in\mathbb{Z}} |\hat{f}(n)|^2 + |\widehat{\partial f}(n)|^2$$

$$= \sum_{n\in\mathbb{Z}} (1+n^2)\, |\hat{f}(n)|^2$$

so $\mathcal{F}$ lands in $\ell^{1,2}$ precisely when $|\partial f| \in L^2$.

Caution 21.6: $L^{k,2} \subseteq L^{k-1,2}$ is a (bounded) inclusion, $\|f\|_{L^{k-1,2}} \leq \|f\|_{L^{k,2}}$

of a dense Banach space. So $L^{k,2}$ is closed in its own norm but dense in the $L^{k-1,2}$-norm.

Thm 21.7: If $f \in L^{1,2}(S^1;\mathbb{C})$, then $f_N \to f$ uniformly.

Proof: It suffices to show $\sup_{S^1} |f| \leq C\|f\|_{L^{1,2}}$

Then, $f_N \to f$ in $L^{1,2}$. so $\sup_{S} |f_N - f| \leq C\|f_N - f\|_{L^{1,2}} \to 0$.

($\hat{f}_N \to \hat{f}$ in $\ell^{1,2}$ and isometry)

And

$$\sup_{S^1} |f| \leq \Big|\sum_n \hat{f}(n)e^{inx}\Big| \leq \sum_{n\in\mathbb{Z}} |\hat{f}(n)|$$

$$\leq \sum_{n\in\mathbb{Z}} \frac{|\hat{f}(n)|}{\sqrt{1+n^2}} \sqrt{(n^2+1)}$$

$$\leq \Big(\sum_{n\in\mathbb{Z}} |\hat{f}|^2(n^2+1)\Big)^{1/2} \Big(\sum_{n\in\mathbb{Z}} \frac{1}{n^2+1}\Big)$$

$$\leq C\|\hat{f}\|_{\ell^{1,2}} = C\|f\|_{L^{1,2}}.$$

Corollary 21.8: If $f \in L^{k,2}(S^1;\mathbb{C})$ then

$$\sup |f_N - f| + \sup|df_N - df| + \dots + \sup|d^{k-1}f_N - d^{k-1}f| \to 0.$$

Rem 21.9: $\ell^{s,2}$ makes sense for non-integer s, which gives a notion of fractional derivatives

$$\left(\frac{d}{d\theta}\right)^s f = \left(\mathcal{F}^{-1} \circ |n|^s \circ \mathcal{F}\right) f.$$

Actually the proof shows any $L^{s,2}$ for $s > \frac{1}{2}$ is okay.

21.iii) Fourier transform of L^p

Recall since S^1 is compact

$$L^2 \subseteq L^p \quad 1 \leq p \leq 2$$
$$\|\!\wr$$
$$\ell^2 \subseteq \ell^q \quad q \geq 2.$$

Thm 21.10 (Hausdorff-Young) Let $1 \leq p \leq 2$, and q be the Hölder conjugate. $\frac{1}{p} + \frac{1}{q} = 1$.

Then $\|\mathcal{F}(f)\|_{\ell^q} \leq \|f\|_{L^p(S^1;\mathbb{C})}$

In particular, $\mathcal{F} : L^p(S^1;\mathbb{C}) \to \ell^q(\mathbb{Z};\mathbb{C})$ is bounded.

Proof: For $p=1$, $q=\infty$.

$$\|\hat{f}\|_\infty = \sup_n \left| \int \langle f, e^{inx} \rangle \right| \leq \|f\|_{L^1}.$$

$p=2$ is obvious. Interpolation or Young Convolution for general.

Ex 21.11: $\{1, 1, \dots\} = \mathcal{F}(\delta)$ in ℓ^∞ but $\delta \notin L^1$ so not surj.

$$\|D_N\|_{L^1} \approx \log N.$$

Thm 21.11: If $f \in L^p(S^1;\mathbb{C})$ then for $1 < p < \infty$,

$$\|f_N - f\|_{L^p} \to 0.$$

Proof: "Convergence of Fourier Series in L^p Space" J. Miao, various Harmonic Analysis texts.

Lecture 22 Fourier Series and PDEs I: Fredholm and Compact Operators

Main Idea 22.1: $\mathcal{F}$ turns differentiation into multiplication

$$\begin{array}{ccc} L^{2,2}(S^1;\mathbb{C}) & \xleftrightarrow{\mathcal{F}} & \ell^{2,2}(\mathbb{Z};\mathbb{C}) \\ -\partial_\theta^2 \downarrow & & \downarrow \ell^2 \\ L^2(S^1;\mathbb{C}) & \xleftrightarrow{\mathcal{F}} & \ell^2(\mathbb{Z};\mathbb{C}) \end{array}$$

Commutes. Eg.

$$L = \sum \alpha_j (-i\partial_\theta)^j f \longleftrightarrow \cdot\left(\sum \alpha_j n^j\right)\cdot \hat{f}$$
$$= (\alpha_N n^N + \alpha_{N-1} n^{N-1} \quad)\ldots$$

$\Rightarrow$ Can solve $Lf = g$ by dividing Fourier transform

$$f = \mathcal{F}^{-1} \circ \frac{1}{|\ell|^N + \alpha|\ell|^{N-1} + \cdots + \alpha_0} \cdot \mathcal{F}g := P_g$$

- As $|\ell| \to \infty$, only leading order matters
- Write $L = (|\ell|^N + \alpha|\ell|^{N-1} + \cdots + \alpha_0)$

$|\ell|^N$: leading order $\simeq$ Invertible; $\alpha|\ell|^{N-1} + \cdots + \alpha_0$: lower order $\simeq$ Compact / f.d.

Invertible + Compact / f.d. = Fredholm.

Question 22.2: Given $f \in \mathcal{H} \subseteq L^2(S^1;\mathbb{C})$, when can we solve

$$Lu = f$$

and what space is u in?

22.i) : Compact Operators

Def 22.3: Let H_1, H_2 be Hilbert spaces. An (bounded) operator

$$K : H_1 \longrightarrow H_2$$

is said to be compact if

$$\{x_n\} \subseteq X \text{ bounded} \Rightarrow \{Kx_n\} \text{ has a convergent subsequence.}$$

i.e. the image of the closed unit has compact closure.

Lemma 22.4: If K has finite rank, then K is compact.

Proof: Let $E \subseteq H_2$ be $\mathrm{Im}(K)$ and choose $E \cong \mathbb{R}^N$ some N.

If $\{x_n\}$ bounded by m,

$$\|Kx_n\| \leq \|K\| \, \|x_n\| \leq \|K\| \, \|m\|.$$

so $\overline{\mathrm{Im}(B_1)}$ is closed and bounded.

Lemma 22.5: If $K_n \longrightarrow K$ i.e. $\forall \varepsilon > 0 \; \exists N$ st $\|(K - K_n)x\| \leq \varepsilon \|x\| \; \forall n \geq N$ and K_n is finite rank, then K is compact.

Proof: Let $\{x_j\}$ be a bounded sequence.

Let $\varepsilon > 0$, choose N so $\|K - K_N\| < \varepsilon/3$, and for that N, let J be large so $\|K_N x_j - K_N x_k\| < \varepsilon/3$ for $j,k \geq J$. Then for j,k

$$\begin{aligned}\|Kx_j - Kx_k\| &\leq \|Kx_j - K_N x_j + K_N x_j - K_N x_k + K_N x_k - K x_k\| \\ &\leq \|Kx_j - K_N x_j\| + \|K_N x_j - K_N x_k\| + \|K_N x_k - K x_k\| \\ &\leq \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3}.\end{aligned}$$

□

Rem 22.6: This is iff (but not in Banach spaces!)

Lemma 22.7: The unit ball $B \subseteq H$ is compact iff H is finite dimensional.

Proof: $\Leftarrow$ is Heine-Borel

$\Rightarrow$ if e_i is an infinite orthonormal basis $\|e_i - e_j\| = \sqrt{2}$ so no conv. subsequence.

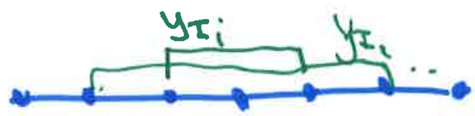
Thm 22.8 (Rellich's Lemma) The inclusion

$$L^{k+1,2}(S^1; \mathbb{C}) \subseteq L^{k,2}(S^1; \mathbb{C})$$

is compact.

Proof: Suffices to prove for $\ell^{1,2} \subseteq \ell^2$.

For each N let



$a_1, a_2, \ldots \to x_1$ conv subsequence for $|\ell| \leq 1$ $\subseteq \mathbb{R}^3$

$b_1\ b_2 \to x_2$ subsubseq for $|\ell| \leq 2$.

Claim diagonal $\{y_n\}$ is Cauchy. 1) For enough N, $\|y_n - y_m\| \leq \varepsilon/2$ on $|\ell| \leq N$

$$= \sum_{\ell=-N}^{N} |y_n - y_m|^2$$

2) $$\sum_{|\ell| \geq N} |y_n^\ell - y_m^\ell|^2 \leq \frac{1}{N} \sum_{|\ell| \geq N} |\ell|^2 |y_n^\ell - y_m^\ell|^2 \leq \frac{2}{N}.$$ $\square$

22.ii: Fredholm Operators

Def 22.9: A bounded operator $L: H_1 \to H_2$ is said to have closed range if $L(H_1) \subseteq H_2$ is a closed subspace.

Lemma 22.10 (closed range estimate) $L: H_1 \to H_2$ has closed range iff $\exists C$ s.t.

$$\|x\|_{H_1} \leq C \|Lx\|_{H_2} \qquad x \perp \operatorname{Ker} L.$$

Proof: $\Rightarrow$ If $Lx_i \to y$ then $\|x_i - x_j\| \leq C \|Lx_i - Lx_j\| \to 0$ so $x_i \to x$ and by continuity $Lx = y$.

$\Leftarrow$ Suppose $\|x_i\| = 1$ w/ $\|Lx_i\| \to 0$; then $\operatorname{Im}(B) \subseteq \operatorname{Ker}(L)^\perp$ (unit ball) is closed, and contains 0, so $\exists x \in B$ w/ $Lx = 0$ $\rightarrow\leftarrow$ $\square$

Def 22.11 : An operator $L: H_1 \to H_2$ is said to be Fredholm if

- L has f.d. kernel
- L has closed range
- L has f.d. cokernel.

Rem 22.12 : Closed range is key! $H_2 / \mathrm{Im}(L) \overset{\text{vs quot}}{=} H_2 / \overline{\mathrm{Im}(L)} \simeq \mathrm{Im}(L)^\perp$ Hilbert space quot.

eg $L^2 / L^{1,2}$ is a bit hard to parse

Thm 22.13 : An operator $L: H_1 \to H_2$ is Fredholm iff
$\exists P: H_2 \to H_1$ st

$$LP = \mathrm{Id} + K_1 \qquad PL = \mathrm{Id} + K_2$$

For $K_i : H_i \to H_i$ compact.

Proof : $\Rightarrow$ Suppose L Fredholm. Let P be inverse of

$$\mathrm{Ker}(L)^\perp \rightleftarrows_P \mathrm{Im}(L)$$ (bounded by open mapping thm)

Then $PL = \mathrm{Id} - \pi_{\mathrm{Ker}\, L}$ which is compact b/c f.d. finite rank
$LP = \mathrm{Id} - \pi_{\mathrm{Im}^\perp}$ "

$\Leftarrow$ $0 = PL(x_i) = x_i - Kx_i$ for $\|x_i\| = 1$ in $\mathrm{Ker}\, L$.
$\|x_i\| \leq \|Kx_i\|$ but converges of x_{i_j} so
x_{i_j} converges $\to B \cap \mathrm{Ker}\, L$ is compact, so f.d.

For closed range:
$$\|x\| \leq \|PLx + Kx\|$$
$$\leq C\|Lx\| + C\|Kx\|$$

Then $LP(y_i) = y_i + Ky_i$ so $y_i = Ky_i$ on $\mathrm{Im}^\perp$. Same.

~~Now suppose $x \perp \mathrm{Ker}\, L$~~ If $Lx_i \to y$ then

$$\|x_i - x_j\| \leq C\left(\|L(x_i) - Lx_j\| + \|Kx_i - Kx_j\|\right)$$ on subseq.

$\to 0$ $\qquad$ 0

Corollary 22.14 : Fredholm implies $\exists C$, K compact so
$$\|x\|_{H^1} \leq C(\|Lx\|_{H^2} + \|Kx\|_{H^1})$$

Lecture 23 Fourier Series and PDEs II: Elliptic Equations

Recall the original goal was to study solutions of

$$Lu = f$$

For differential operators L.

Question 23.1: For what f can this be solved?
What is $\mathrm{Ker}(L) \subseteq L^{k,2}$?

Question 23.2: For $f \in L^{k,2}$, what space does u lie in?

Recall $\Delta = -\partial_\theta^2 = (-i\partial_\theta)^2$

We will consider four examples

i) $L = \Delta + 1$

ii) $L = \Delta$

iii) $L = \sum_{j=0}^{d} \alpha_j (-i\partial_\theta)^j \quad \alpha_i \in \mathbb{C}.$

iv) $L = \Delta + V(\theta)$ (time-ind Schrödinger)

Def 23.3: An operator $L = \sum_{j=0}^{N} \alpha_j(x)(-i\partial_\theta)^j$ is called a differential operator for $\alpha_j : S^1 \to \mathbb{C} \in C^\infty$.
It is said to be elliptic if $|\alpha_N(x)| > 0$.

Observe that an N^{th} order operator is a bounded map

$$L : L^{k+N,2}(S^1;\mathbb{C}) \longrightarrow L^{k,2}(S^1;\mathbb{C}).$$

since $\|Lu\|_{L^2} \leq \|\sum_{j=0}^{N} \alpha_j(x)(-i\partial_\theta)^j u\|_{L^2} \leq \sum_{j=0}^{N} \|\partial_\theta^j u\|_2 \leq \|u\|_{L^{N,2}}$

same for $L^{k,2}$.

23.i) $L = \Delta + 1$

$$\begin{array}{ccc} L^{2,2}(S^1;\mathbb{C}) & \overset{\mathcal{F}}{\longleftrightarrow} & \ell^{2,2}(S^1;\mathbb{C}) \\ \Delta+1 \big\downarrow & & \big\downarrow |e|^2+1 \\ L^2(S^1;\mathbb{C}) & \overset{\mathcal{F}}{\longleftrightarrow} & \ell^2(S^1;\mathbb{C}) \end{array}$$

Prop 23.4: $\Delta + 1 : L^{2,2}(S^1;\mathbb{C}) \to L^2(S^1;\mathbb{C})$ is an isomorphism

Proof: Suffices to show $|e|^2+1 : \ell^{2,2} \to \ell^2$ is, but this obviously has inverse $\frac{1}{|e|^2+1}$.

To be pedantic,

$$\left\| \frac{1}{|\ell|^2+1} a_\ell e_\ell \right\|_{\ell^{2,2}} = \sum_{\ell\in\mathbb{Z}} (1+|\ell|^2+|\ell|^4) \cdot \frac{|a_\ell|^2}{(1+|\ell|^2)^2}$$

$$= \sum \frac{1+|\ell|^2+|\ell|^4}{1+2|\ell|^2+|\ell|^4} |a_\ell|^2 \leq C \|a_\ell\|_2.$$

Corollary 23.5: 1) $\Delta+1 : L^{k+2,2}(S^1;\mathbb{C}) \to L^{k,2}(S^1;\mathbb{C})$ is an isomorphism $\forall k\in\mathbb{N}$.

2) $\Delta+1$ (a fortiori) is Fredholm $\forall k$,

3) $\Delta+1$ satisfies

$$\|u\|_{L^{k+2,2}} \leq C_k \|(\Delta+1)u\|_{L^{k,2}} \quad \text{for } C_k \text{ constants.}$$

4) $Lu=f$ has a solution u, $\forall f\in L^{k,2}$ and $u\in L^{k+2,2}$.

Proof: Only 1) needs proof

$$(1+|\ell|^2+\ldots+|\ell|^{2k}+|\ell|^{2k+2}+|\ell|^{2k+4}) \sim 1+|\ell|^{2k+4}$$
$$\sim (1+|\ell|^{2k})(1+|\ell|^2)^2$$
□

Remark 23.6: Actually the same result holds for $s\in\mathbb{R}$.

23.ii) $L=\Delta$.

Def 23.7: The (formal) adjoint of an operator L is that L^* such that

$$\langle Lu, v\rangle_{L^2} = \langle u, L^*v\rangle_{L^2}$$

$\forall u\in C^\infty$. L is said to be formally self-adjoint if $L=L^*$.

Ex 23.8: $\langle Lu,v\rangle = \int \sum \langle \alpha_j(x)(\partial_\theta)^j u, v\rangle$

$$= \int \langle u, (-i\partial_j)^{j} \bar{\alpha}_j(x) v\rangle$$

$$= \bar{\alpha}_N(x)(-i\partial_\theta)^N + \bar{\alpha}_{N-1}(-i\partial_\theta)^{N-1} + (\partial\alpha_N)(-i\partial_\theta)^{N-1} + \ldots$$

Ex 23.9 Δ is formally self adjoint.

$$\int \langle -\partial_\theta^2 u, v\rangle = \int \langle -\partial_\theta u, -\partial_\theta v\rangle = \int \langle u, -\partial_\theta^2 v\rangle.$$

Thm 23.10: The operator

$$\Delta : L^{k+2,2}(S^1;\mathbb{C}) \longrightarrow L^{k,2}(S^1;\mathbb{C})$$

is Fredholm. It has 1-dimensional kernel and cokernel, given by the constants.

Proof 1: On Fourier space $\Delta = |e|^2$. Set $P = \begin{cases} \frac{1}{|e|^2} & e \neq 0 \\ 0 & e = 0 \end{cases}$.

Then $P\Delta = \mathrm{Id} - \pi_0$

$\Delta P = \mathrm{Id} - \pi_0$ and π_0 has rank 1, so compact.

Therefore, Δ is Fredholm. Clearly constants are both the kernel and cokernel.

~~Proof 2 (of cokernel)~~) **Corollary 23.11** (Elliptic estimates)

$$\|u\|_{L^{2,2}} \leq \|\Delta u\|_{L^2} + \|\pi_0 u\|_{L^2} \leq \|\Delta u\|_{L^2} + \|u\|_{L^2}$$

$$\|u\|_{L^{k+2,2}} \leq \|\Delta u\|_{L^{k,2}} + \|u\|_{L^{k,2}}$$

Corollary 23.12 (Elliptic regularity). If $\Delta u = f$ w/ $f \in \mathcal{C}^\infty$ (in particular 0)

then $u \in L^2 \Rightarrow u \in L^{2,2} \Rightarrow u \in L^{4,2} \dots$

$\Rightarrow u \in \mathcal{C}^\infty$.

Proof: $\|u\|_{L^{k+2,2}} \leq \|\Delta u\| + \|u\|_{L^{k,2}}$ same for f.

Remark 23.13: "u is 2 degrees more regular than itself"

This is called "elliptic bootstrapping".

Proof 2 (of cokernel): If $v \in \mathrm{Coker}$, then

$0 = \langle \Delta u, v\rangle = \langle u, \Delta v\rangle$ for all $u \Rightarrow \Delta v = 0$ and so $v \in L^{2,2}$.

Therefore $v \in \mathrm{Ker}(\Delta)$, so constant.

iii) General Const Coefficient

Thm 23.14: $L = \sum_{j=1}^{N} a_j (-i\partial_t)^j$ is Fredholm. It satisfies

$$\|u\|_{L^{k+2,2}} \le C_k \|Lu\|_{L^{k,2}} + \|u\|_{L^{k,2}}$$

and elliptic regularity. It has dim Ker = dim Coker = {# integer roots of $\sum a_j \lambda^j = 0$}

Proof: $\mathcal{F}^{-1} L \mathcal{F} = a_0 + a_1 \ell + a_2 \ell^2 + \dots + a_N \ell^N$.

Rem 23.15: Agrees w/ ODE theory that N^{th} order has $\le N$ solutions.

iv) The time-independent Schrödinger Equation

Say $V(\theta) \in C^\infty$ w/ $V \ge 0$.

$$\Delta + V \longleftrightarrow |\ell|^2 + \hat{V} *.$$ cw.

Thm 23.16: $\Delta + V$ is Fredholm and formally-self adjoint. $\exists C_k$ such that

$$\|u\|_{L^{k+1,2}} \le C_k \|(\Delta+V)u\|_{L^{k,2}} + \|u\|_{L^{k,2}}$$

hold.

Proof:

$$\|u\|_{L^{k+2,2}} \le \|\Delta u\|_{L^{k,2}} + \|u\|_{L^{k,2}}$$
$$\le_{C_k} \|(\Delta + V - V)u\|_{L^{k,2}} + \|u\|_{L^{k,2}}$$
$$\le_{C_k} \|(\Delta+V)u\|_{L^{k,2}} + \|u\|_{L^{k,2}} \qquad C_k \sim \|V\|_{C^k} \quad \square$$

Thm 23.17: In fact, $\operatorname{Ker}(\Delta+V) = 0$.

~~If not,~~ Can show $\|u\|_{L^2} \le \|\Delta + Vu\|_{L^2}$ then $\|u\|_{L^{k+2}} \le C_k \|(\Delta+V)u\|_{L^{k}}$

If not $\exists u_n$ st $\|u_n\|_{L^2} = 1$, but $\downarrow \le \frac{1}{n}$.

Then 1) $\langle (\Delta+V)u, u\rangle \le \dfrac{\frac{1}{\sqrt{n}}\|u_n\|^2_{L^2}}{2} + \dfrac{\sqrt{n}\|(\Delta+V)u_n\|^2_{L^2}}{2} \to 0$

$$\|\nabla u_n\|^2_{L^2} + \langle u, Vu\rangle_{L^2} \longrightarrow 0.$$

2) $u_n \in L^2$, so $\|u_n\|_{L^{2,2}} \leq \frac{1}{n} + \|u_n\|_{L^2} < \infty$.
$\leq C_{\infty 2}$ bounded

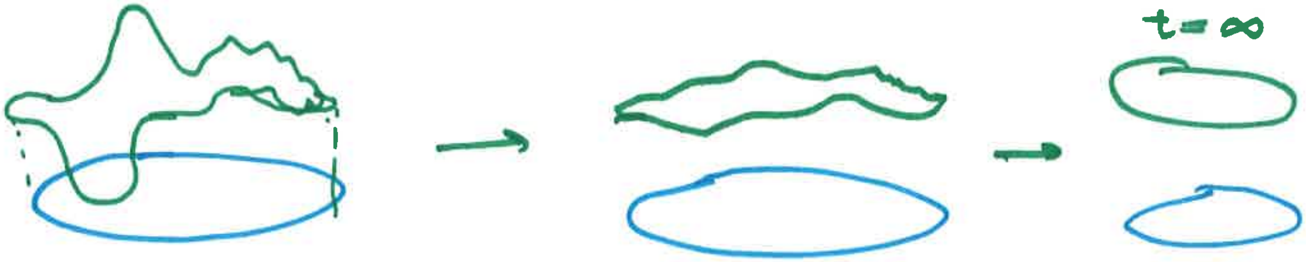
$\Rightarrow u_{n_k} \in L^{1,2}$ convergent, but $\|\nabla u_{n_k}\| \to 0$ so $u_\infty = $ const,
$\to u_\infty$
since $\|u_n\|=1$,
$u_\infty = \frac{1}{2\pi}$.

but $\langle u, Vu \rangle = \frac{1}{8\pi^3} \int V > 0$. $\rightarrow\leftarrow$

$\square$.

Lecture 24 Fourier Series and PDEs III: the heat equation and heat kernel.

Question 24.1
Suppose that $f(x) \in C^\infty(S^1 : \mathbb{R})$ is a smooth heat distribution at time $t=0$. What is the distribution of heat $u(x,t)$ for all t?



Def 24.2: the initial value problem for the heat equation is to solve

$$\begin{cases} (\partial_t + \Delta_x) u(x,t) = 0 = g(x,t) \text{ heat source} \\ u(x,0) = f(x). \end{cases}$$

for $t \geq 0$.

Ex 24.3: Suppose that $f(x) = \cos(nx)$. Then

$$\Delta_x f = -\partial_x^2 f = n^2 f \quad \text{so}$$

$$(\partial_t + \Delta_x) e^{-n^2 t} \cos(nx) = (-n^2 + n^2) e^{-n^2 t} \cos(nx) = 0.$$

is a solution.

Thm 24.4 (general solution) The heat evolution preserves Fourier modes and if

$$f(x) = \sum_{k \in \mathbb{Z}} a_k \cos(kx) + b_k \sin(kx)$$

then

$$u(x,t) = \sum_{k \in \mathbb{Z}} e^{-k^2 t} \left(a_k \cos(kx) + b_k \sin(kx) \right)$$

solves

$$\begin{cases} (\partial_t + \Delta_x) u(x,t) = 0 \\ u(x,0) = f(x). \end{cases}$$

Lemma 24.5: If $f(x) \in L^2(S^1;\mathbb{R})$ then $u(x,t) \in C^\infty(S^1;\mathbb{R})$ for each $t>0$.

Proof: It suffices to show that $\sum |u_k|^2 |k|^{2s} < \infty$ for all s.
But $u_k = e^{-k^2 t}(a_k + i b_k)$ so $\sum (|a_k|^2 + |b_k|^2)|k|^{2s} e^{-k^2 t} \le C \sum |a_k|^2 + |b_k|^2$
because $|k|^{2s} e^{-k^2 t} < 1$ for k large. $\le C\|f\|_{L^2}$.

Proof (of thm 24.4)

By the above $\|u\|_{L^{2,2}} < \infty$, so

$$\Delta_x \Big(\underbrace{\sum_{|k|=1}^{N} e^{-k^2 t} (a_k \cos(kx) + b_k \sin(kx))}_{u_N} \Big) \longmapsto \Delta u.$$

For each $t_* \in [t_0, t_1]$ the bound is uniform,

$$\partial_t \Big(\sum_{|k|=1}^{N} e^{-k^2 t} (\qquad) \Big) \longrightarrow \partial_t u \quad \text{uniformly}$$

(use large s, so bringing down k^2 doesn't matter).
Then by taking limits

$$(\partial_t + \Delta_x) u = (\partial_t + \Delta_x) u_N = 0 \quad \text{by previous example.}$$

Obviously $u(x,0) = f$ □

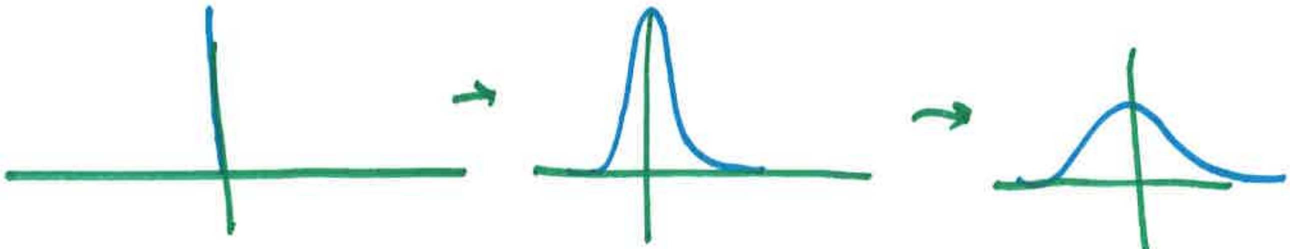
Corollary 24.6: $u(x,t) \in C^\infty([t_0,\infty) \times S^1;\mathbb{R})$ for any t_0.

Proof: Lemma 24.5 shows x derivatives exist.
For t derivatives, $k^{2s} \cdot u_N \longrightarrow u$ and
$k^{2s} \cdot u_N' \longrightarrow v$ uniformly
for any s, thus $\partial_t^s u = \partial_t^s u_N = k^{2s} \cdot u_N$.

Mixed partials same. □

24.ii) The Idea of a fundamental solution

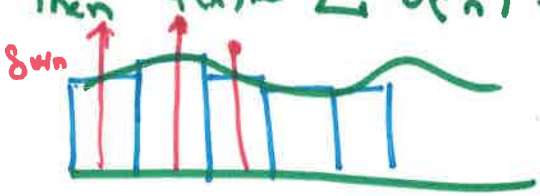
Consider a pinprick of heat



Observation 24.7: Suppose we can find the solution ~~Question~~

of
$$\begin{cases} (\partial_t + \Delta_x) k(x,t) = 0 \\ u(x,0) = \delta_x. \end{cases}$$

Then $f(x) \approx \sum_i \delta(x-\frac{i}{n}) f(\frac{i}{n}) \longrightarrow \int_0^{2\pi} \delta(x-y) f(y)\,dy.$



the heat equation is Linear so

$$u_f(x,t) = \sum_{j=1}^{N} K\left(x-\frac{j}{n},t\right)\cdot f\left(\frac{j}{n}\right). \longrightarrow \int_0^{2\pi} K(x-y,t) f(y)\,dy.$$

Def 24.8: the solution $K(x-y,t) : S^1 \times S^1 \times \mathbb{R}^{\geq 0} \to \mathbb{R}$ is called the heat kernel or fundamental solution.

Thm 24.9: There exists a heat kernel $K(x,t)$ such that

$$(\partial_t + \Delta_x) K(x,t) = 0 \quad \forall t > 0$$

and as $t \to 0$,

- $\int_0^{2\pi} K(x,t)\,dx \to 1$ as $t \to 0$
- $\int_\delta^{2\pi-\delta} K(x,t)\,dx < \varepsilon$, $\forall \varepsilon > 0, \delta > 0$, $\exists t$ such that
- $\int_0^{2\pi} K(x-y,t) f(y)\,dy \longrightarrow f(x)$ uniformly as $t \to 0$.

Proof: Recall the Dirichlet Kernel

$$D_N(x) = \Re\left(\sum_{j=-N}^{N} e^{in\theta}\right).$$

thus ie $\hat{D}_N = \{1\}_{k\in\mathbb{Z}}$.

Thus set

$$K(z,t) = \sum_{k\in\mathbb{Z}} e^{-k^2 t}\cos(kz).$$

By the same logic as before, $K(x,t) \in C^\infty((0,\infty)\times S^1;\mathbb{R})$ and

$$(\partial_t + \Delta_x)K(x,t) = 0.$$

Lemma 24.10 (Poisson Summation) ← proved next week.

$$\sum_{k\in\mathbb{Z}} f(x+k) = \sum_{n\in\mathbb{Z}} e^{in\theta}\hat{f}(n).$$

In particular, since $\mathcal{F}(e^{-ax^2}) = e^{-x^2/a}$, the following results

$$K(z,t) = \sum_{k\in\mathbb{Z}} e^{-k^2 t}\cos(kz) = \sum_{k\in\mathbb{Z}} \frac{1}{\sqrt{4\pi t}}\, e^{-\frac{(z-k)^2}{4t}}$$

bullet points 1), 2) obvious from Fourier, real space side resp.
For 3) know D_N satisfies this, and $K_N \to D_N$ uniformly. □

Rem 24.11: Kernels are used as models for the linear parabolic theory underlying "flow" equations e.g. Mean Curvature flow, Ricci Flow, Yang-Mills Flow.

It's not a coincidence that $\int K(x,y,t)f(y)\,dy \xrightarrow{t\to\infty}$

$$\lim_{t\to\infty} \mathrm{Tr}\left(f \mapsto \int K(x-y,t)f(y)\right) = 1 = \dim \ker \Delta$$

This is used in Getzler's famous Heat kernel proof of the Atiyah-Singer Index theorem.

Lecture 25 Fourier Transforms I: Schwartz functions + distributions.

Question 25.1: What is the analogue of Fourier series on $\mathbb{R}$?

Recall functions on $\mathbb{R}$ are more subtle b/c we must control the behavior as $|x| \to \infty$.

On $[-\pi, \pi]$, write $f(x) = \sum_k a_k e^{ikx}$ periodic functions

$[-T\pi, T\pi]$ $\quad f(x) = \sum_k a_k e^{i\frac{k}{T}x}$ for $\frac{k}{T} = 0, \frac{1}{T}, \frac{2}{T}$.

Note the discretization of the series gets finer as $T \to \infty$. Write $\xi_k = \frac{k}{T}$ then

$\hat{f}$ space

$$g(\xi_k) = \frac{1}{2\pi}\int_{-T/2}^{T/2} f(x)\, e^{i\xi_k x} \quad \text{and}$$

$$f(x) = \frac{1}{2\pi T}\sum g(\xi_k) e^{ix\xi_k} = \sum g(\xi_k) e^{i\xi}(\xi_{k+1} - \xi_k)$$

$$\to \frac{1}{2\pi}\int_{-T/2}^{T/2} e^{i\xi x} g(\xi)\, d\xi$$

In limit, $g = \hat{f}$ becomes a function of continuous variable ξ.

Rem 25.2: Fourier series has nice properties

Series		Transform	or Spectral thing
1) Countable basis		✗	✓
2) $-i\partial_x \longleftrightarrow k$.	$\Longleftrightarrow$	✓	✗
3) $\|f(x)\|_{L^2} = \|\hat{f}\|_{L^2}$		✓	✗

Observation 25.3: It is no longer clear $\hat{f}$ is integrable, or $\mathcal{F}, \mathcal{F}^{-1}$ make sense etc.

Def 25.~~3~~4: Given f in some function space X, define the Fourier transform by

$$\hat{f} = \mathcal{F}(f) = \frac{1}{2\pi}\int_{\mathbb{R}} f(x)\, e^{i\xi x}\, dx$$

and the inverse transform by

$$g = \mathcal{F}^{-1}(\hat{f}) = \frac{1}{2\pi}\int_{\mathbb{R}} \hat{f}(\xi)\, e^{-i\xi x}\, d\xi.$$

25.i: Schwartz Functions

Def 25.5: A function is said to be Schwartz if for all $\alpha \in \mathbb{N}$, $N \in \mathbb{N}$

$$|x^N \partial_x^\alpha f| \leq C_{N,\alpha}$$

i.e. functions whose derivatives all decay faster than polynomially.

Ex 25.6: e^{-x^2} is schwartz.

Prop 25.7: $\mathcal{S}(\mathbb{R})$ the Schwartz functions satisfies

1) is a vector space and algebra
2) $\partial_x, \int_{-\infty}^x : \mathcal{S}(\mathbb{R}) \to \mathcal{S}(\mathbb{R})$, and $\cdot e^{i\xi x} : \mathcal{S}(\mathbb{R}) \to \mathcal{S}(\mathbb{R})$ $\forall \xi \in \hat{\mathbb{R}}$
3) ~~obviously~~ $\mathcal{S}(\mathbb{R}) \subseteq L^\infty(\mathbb{R}) \cap L^2(\mathbb{R}) \cap L^1(\mathbb{R})$.

Proof: these are all obvious.

Warning 25.8: $\mathcal{S}(\mathbb{R})$ is not a Banach or Hilbert space. The topology is a bit more complicated, and stronger.

$$\varphi_n \to \varphi \quad \text{iff} \quad |x^N \partial_x^\alpha (\varphi_n - \varphi)| \to 0 \ \forall N, \alpha$$

It is what called a (tame) Frechet space w/ a countable filtration of norms.

Lemma/Def 25.9: For each $\varphi \in \mathcal{S}(\mathbb{R})$, $\hat{\varphi}$ is defined and $\hat{\varphi} \in L^\infty(\mathbb{R})$.

$$\hat{\varphi}(\xi) = \frac{1}{2\pi} \int_{\mathbb{R}} f(x) e^{i\xi x}\, dx \leq \frac{1}{2\pi} \int_{\mathbb{R}} |f|\, dx = \|f\|_{L^1} < \infty.$$

So take sup.

25.ii) Properties of the Fourier transform

Prop 25.10: The Fourier transform on $\mathcal{S}(\mathbb{R})$ obeys

1) $\widehat{f(x+y)} = e^{-i\xi y} \hat{f}(\xi)$, $\quad \widehat{e^{ix\zeta} f(x)} = \hat{f}(\xi + \zeta)$
2) $\widehat{\partial_x^k f} = (-i\xi)^k \hat{f}$

3) $\widehat{(ix)^k f(x)} = \partial_\xi^k \hat{f}(\xi)$.

4) $\widehat{f * g} = \hat{f} \cdot \hat{g}$ (inverse hard to show directly).

Proof:

1) $\widehat{f(x+y)} = \int_{\mathbb{R}^n} f(x+y) e^{i\xi x} dx = \int_{\mathbb{R}^n} f(x) e^{i(x-y)\xi} dx = e^{-iy\xi} \hat{f}(\xi)$.

2) $\widehat{(\partial_x^k) f} = \int_{\mathbb{R}^n} (\partial_x^k f) e^{i\xi x} dx$

$= (-1)^k \int_{\mathbb{R}^n} f \, \partial_x^k (e^{i\xi x}) dx$ Int by parts.

$= (-i\xi)^k \int_{\mathbb{R}^n} f e^{i\xi x} dx$

3) $\frac{\partial}{\partial \xi} \hat{f} = \lim_{h \to 0} \int_{\mathbb{R}} f(x) \frac{e^{i(\xi+h)x} - e^{i\xi}}{h} dx$ ↓ dominated convergence

$= \int_{\mathbb{R}} f(x) (ix) e^{i\xi x} dx$

$= \widehat{(ix) f(x)}$ and induct.

4) same as series.

<u>Corollary 25.11</u>: $\mathcal{F} : \mathcal{S}(\mathbb{R}) \to \mathcal{S}(\mathbb{R})$
$\mathcal{F}^{-1} : \mathcal{S}(\mathbb{R}) \to \mathcal{S}(\mathbb{R})$

Proof: As before, if $\varphi \in \mathcal{S}(\mathbb{R})$, $|\hat{\varphi}| < \infty$.
and $|\xi^N \partial_\xi^k \hat{\varphi}| = |\widehat{x^k \partial_x^N \varphi}| < \infty$ b/c $x^k \partial_x^N \varphi \in \mathcal{S}(\mathbb{R})$
The same properties hold w/ $\mathcal{F}^{-1}$ but w/ $(-1)^{N+k}$. □

25. iii) Fourier Inversion I

Thm 25.12 : $\mathcal{F} : \mathcal{S}(\mathbb{R}) \to \mathcal{S}(\mathbb{R})$ is an isomorphism with inverse $\mathcal{F}^{-1}$.

Proof : $\mathcal{F}^{-1} \circ \mathcal{F}(x^N \partial_x^k \varphi) = \mathcal{F}^{-1}(\partial_\xi^N \xi^k \mathcal{F}(\varphi))$
$= x^N \partial_x^k \mathcal{F}^{-1}\mathcal{F}(\varphi)$.

Thus $T = \mathcal{F}^{-1} \circ \mathcal{F} : \mathcal{S} \to \mathcal{S}$ commutes w/ x, ∂_x.

Lemma 25.13 : If $[T, x]$, $[T, \partial_x] = 0$ on $\mathcal{S}$, then $T = c\,\mathrm{Id}$ for $c \in \mathbb{R}$.

Proof: **Step 1** : If $\varphi(y) = 0$ then $T\varphi(y) = 0$.

$\varphi = (x-y)\varphi_1$ w/ $\varphi_1 \in \mathcal{S}$ by Taylor's theorem.

Thus $T\varphi(y) = T(x-y\,\varphi_1)(y)$
$= [(x-y)T\varphi_1](y) = 0$.

Step 2 : Fix $y \in \mathbb{R}$, g st $g(y) = 1$ set $c(y) = (Tg)(y)$.

Then $Tf(y) = c(y)f(y)$. To see this,

$\varphi_2 = f(x) - f(y)g(x)$ so $\varphi_2(y) = 0$,

so $0 = T\varphi_2 = Tf(y) - f(y)(Tg)(y)$
$= Tf(y) - f(y)c(y)$ so $Tf(y) = c(y)f(y)$ $\forall y$.

Step 3 : Take $f \neq 0$ eg e^{-x^2}, then $c(y) = \frac{Tf}{f} \in \mathcal{S}(\mathbb{R})$.

Step 4 : $c(y)(\partial_x f)_y = T(\partial_x f_y) = \partial_x Tf(y)$
$= \partial_x [c \cdot f](y)$.
$= (\partial_x c)_y \cdot f(y) + c(y)\partial_x f(y)$

thus $(\partial_x c)(y) = 0$ $\forall y \Rightarrow c$ is const.

Step 5 : By normalizing conventions, can take $c = 1$.

Lecture 26 Fourier Transforms II: Distributions and Planchurel.

Recall that if $E \subseteq \mathbb{R}^n$ is compact

$$L^\infty(E) \subseteq \dots \subseteq L^4(E) \subseteq L^3(E) \subseteq L^2(E) \subseteq L^1(E)$$

$\Downarrow$ * dual

$$\dots \supseteq L^{4/3}(E) \supseteq L^{3/2}(E) \supseteq L^2(E) \supseteq L^\infty(E)$$

Thus in general $X \subseteq Y \Rightarrow Y^* \subseteq X^*$ (all functionals bounded on Y are automatically bounded on X, but maybe more).

Def 26.1: The space of tempered distributions, denoted $S'(\mathbb{R}) = S(\mathbb{R})^*$ is $\xi : S(\mathbb{R}) \to \mathbb{R}$ which are bounded in the sense $\exists C, N, K$ st

$$|\xi(\varphi)| \leq C \sum_{\substack{k,n \\ \leq K,N}} \sup_{\mathbb{R}} |x^n \partial_x^k \varphi|$$

Ex 26.2: $\alpha = |x|^N$. Then

$$\alpha \text{ "=" } \int_{\mathbb{R}} \alpha \cdot \varphi \, dx = \langle \alpha, \varphi \rangle \text{ is a tempered distribution as}$$

$$\int \alpha \varphi = \int |x|^N \varphi \leq \sup \left| |x|^N \varphi \right| \quad \text{so } N = N.$$

Ex 26.3: $|x|^n \partial_x^k = \beta$

$$\varphi \mapsto \int_{\mathbb{R}} |x|^n \partial_x^k \beta \, dx \quad \text{for same reason}$$

$\Rightarrow$ functions w/ growth slower than some polynomial.

Ex 26.4: Let $\delta \in S'(\mathbb{R})$ be the distribution given by

$$\langle \delta, \varphi \rangle \text{ "=" } \int_{\mathbb{R}} \delta(x) \varphi(x) \, dx = \varphi(0).$$

Def 26.5: δ is called the Dirac delta "function" or distribution.

26.ii) Operators on distributions

Suppose $A : \mathcal{S}(\mathbb{R}) \to \mathcal{S}(\mathbb{R})$ is a linear (continuous) operator.

Def 26.6: The formal adjoint is the operator on distributions

$$A^* : \mathcal{S}(\mathbb{R}) \to \mathbb{R} \text{ given by}$$

$$\langle A^* u, \varphi \rangle := \langle u, A\varphi \rangle.$$

Lemma 26.7: $\forall u \in \mathcal{S}'(\mathbb{R})$, $\exists \varphi_n \in \mathcal{S}(\mathbb{R})$ s.t.

$$\int_{\mathbb{R}} \langle \varphi_n, \varphi \rangle \, dx = \langle \varphi_n, \varphi \rangle \longrightarrow u(\varphi) \quad \text{for all } \varphi.$$

Proof: Take $\varphi_n(x) = \int K_{1/n}(x-y)\, u(y)\, dy$
$= \langle K_{1/n}(x-y), u \rangle$

and apply approximate identity stuff.

Def 26.8: A^* is said to extend B if

$$\langle A^* u, \varphi \rangle = \langle u, A\varphi \rangle = \langle Bu, \varphi \rangle$$

for all $u, \varphi \in \mathcal{S}^*(\mathbb{R})$

Ex 26.9: The distributional derivative of u is

$$\langle \partial_x u, \varphi \rangle := \langle u, -\partial_x \varphi \rangle.$$

Obviously, if $u \in \mathcal{S}(\mathbb{R})$ this holds by integration by parts, so the extension is valid.

Ex 26.10: $H(x) = \begin{cases} 1 & x \geq 0 \\ 0 & x < 0 \end{cases}$. $\quad -\delta = \partial_x H$. Since $\forall \varphi$,

$$\langle H, \partial_x \varphi \rangle = \int_0^\infty \partial_x \varphi = \varphi(\infty) - \varphi(0) = -\varphi(0)$$

26.iii) The Plancharel Theorem

Let $A : \mathcal{S}(\mathbb{R}) \to \mathcal{S}(\mathbb{R})$ be the Fourier transform. Thus to extend to $\mathcal{S}'(\mathbb{R})$ we need a $B : \mathcal{S}(\mathbb{R}) \to \mathcal{S}(\mathbb{R})$ satisfying

$$\int_{\mathbb{R}} (B\psi)\,\varphi(x)\,dx = \langle B\psi, \varphi\rangle = \langle \psi, A\varphi\rangle = \int_{\mathbb{R}} \psi(x)\,\hat{\varphi}(x)\,dx$$

__Thm 26.11__ (Plancharel): $B = \mathcal{F}$. ie, the Fourier transform is self-adjoint on $\mathcal{S}$. ie.

$$\int_{\mathbb{R}} \hat{\psi}\varphi\,dx = \int \psi\hat{\varphi}\,dx$$

__Proof__:

$$\begin{aligned}\int_{\mathbb{R}} \hat{\psi}\varphi\,dx &= \int_{\mathbb{R}}\int_{\mathbb{R}} \psi(\xi)e^{ix\xi}\,d\xi\,\varphi(x)\,dx \\ &= \int_{\mathbb{R}}\int_{\mathbb{R}} \psi(\xi)\,e^{ix\xi}\varphi(x)\,dx\,d\xi \quad \text{(Fubini)} \\ &= \int_{\mathbb{R}} \psi(\xi)\,\hat{\varphi}(\xi)\,d\xi.\end{aligned}$$

□

__Corollary 26.12__: $\mathcal{F}$ extends to a map $\mathcal{F} : \mathcal{S}'(\mathbb{R}^n) \to \mathcal{S}'(\mathbb{R}^n)$ as does $\mathcal{F}^{-1}$, and they remain inverses.

__Proof__: Follows directly, and extension of id is obviously id

__Corollary 26.13__: $\mathcal{F}$ extends to an isomorphism on L^2, in fact an isometry.

So

$$\begin{array}{ccccc} \mathcal{S}(\mathbb{R}) & \hookrightarrow & L^2(\mathbb{R}) & \hookrightarrow & \mathcal{S}'(\mathbb{R}) \\ \downarrow \mathcal{F} & & \downarrow \mathcal{F} & & \downarrow \mathcal{F} \\ \mathcal{S}(\mathbb{R}) & \hookrightarrow & L^2(\mathbb{R}) & \hookrightarrow & \mathcal{S}'(\mathbb{R}) \end{array}$$

commutes. The Plancharel Formula $\|f\|_{L^2} = \|\hat{f}\|_{L^2}$ holds.

Proof: $\mathcal{F}(f)$ is defined as a distribution, so suffices to show $\|\mathcal{F}(f)\|_{L^2} < \infty$. Take $\psi = \overline{\hat{\varphi}}$. Then **26.4**

$$\hat{\psi} = \int \overline{\varphi} e^{-i\xi x} dx = \mathcal{F}^{-1}(\overline{\varphi})$$

so $\hat{\hat{\psi}} = \mathcal{F}\mathcal{F}^{-1}(\overline{\varphi}) = \overline{\varphi}$. Conclude

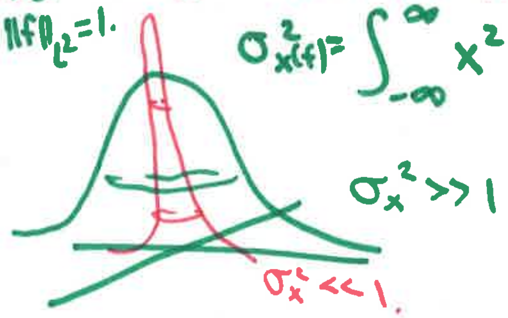
$$\int |\hat{\varphi}|^2 dx = \int |\varphi|^2 dx$$ ∎.

26.iv) The Uncertainty Principle

Let $f \in L^2$. Then the variance of $|f|$ is

w $\|f\|_{L^2} = 1$.

$$\sigma_{x(f)}^2 = \int_{-\infty}^{\infty} x^2 |f|^2 dx - \left(\int_{-\infty}^{\infty} x |f|^2 dx\right)^2$$ ↙ mean



$$\sigma_\xi^2 = \sigma_x^2(\hat{f}).$$

Thm 26.14 (Heisenberg Uncertainty Principle)

$$\sigma_x \sigma_\xi \geq \frac{1}{2}.$$

Proof: may assume mean 0 by translation.

Let $\varphi = xf$, $\hat{\psi} = \xi \hat{f}$. so $\psi = -i\partial_x f$.

By Cauchy Schwartz,

$$|\langle \varphi, \psi \rangle|^2 \leq \left(\int x^2 f^2\right)\left(\int \xi^2 \hat{f}^2\right) = \sigma_x^2 \sigma_\xi^2.$$

But $$|\langle \varphi, \psi\rangle|^2 \geq \mathrm{Im} \geq \left|\frac{\langle \varphi, \psi\rangle - \langle \psi, \varphi\rangle}{2i}\right|^2.$$ $= \left|\frac{i}{2i}\right|^2 = \frac{1}{4}$. ∎.

And $$-\langle \varphi, \psi\rangle + \langle \psi, \varphi\rangle = \int x\overline{f}(-i\partial_x f) - xf(i\partial_x \overline{f})\, dx$$
$$= \int x\overline{f}(-i\partial_x f) + i\partial_x(xf)\overline{f}\, dx$$
$$= \int x\overline{f}(-i\partial_x f) + if\overline{f} + ix\partial_x f\,\overline{f}\, dx = i\int |f|^2 = i$$

Lecture 27 Fourier Transforms III : PDEs and Convolutions

Recall that on $L^2(S^1 : \mathbb{C})$, $\mathcal{F}$ turned Differential operators into multiplication.

Question 27.1 : How do we extend the analysis on S^1 to $\mathbb{R}$?

How do we solve e.g.

$$\Delta u = f \in L^2(\mathbb{R})$$

$$(\Delta + 1)u = f$$

$$Lu = \sum_j \alpha_j(x) \partial_x^j u = f \in L^2(\mathbb{R})$$

or parabolic
$$\begin{cases} (\partial_t + \Delta_x) u(x,t) = g \\ u(x,0) = f \end{cases}$$

and can we extend the results of e.g. elliptic regularity?

Observation 27.2

$$\begin{array}{ccc} L^{2,2}(\mathbb{R}) & \overset{\mathcal{F}}{\longleftrightarrow} & L^2(\mathbb{R}, (1+|\xi|^2)^2 d\xi) \\ \downarrow L & & \downarrow \hat{L} \\ L^2(\mathbb{R}) & \overset{\mathcal{F}}{\longleftrightarrow} & L^2(\mathbb{R}) \end{array}$$

Note eg $\partial_x f \in L^2 \iff \int |\xi f|^2 d\xi = \int |f|^2 \underbrace{|\xi|^2 d\xi} < \infty$.

Thus to solve, one must study the "inverse operator"

$$\hat{f} \longrightarrow \frac{1}{\alpha_0 + \alpha_1 \xi + \dots + \alpha_N \xi^N}$$

(for $\alpha_j \in \mathbb{C}$ constant coefficient).

Def 27.3 : Let $K(z) := \mathcal{F}\left(\frac{1}{\alpha_0 + \dots + \alpha_N \xi^N}\right)$.

Then if $Lu = f$, $\Rightarrow u = \int_{\mathbb{R}} K(x-y) f(y)\, dy = \mathcal{F}^{-1}\left(\frac{1}{\alpha_0 + \dots + \alpha_N \xi^N} \cdot \hat{f}\right)$.

$K(z)$ is called by various authors :
1) the Fundamental solution
2) the Green's function (of L)
3) the integral Kernel
4) the Schwartz Kernel.

Rem 27.4:
$$\mathcal{F}(\delta)(\varphi) = \langle \delta, \mathcal{F}\varphi \rangle = \hat{\varphi}(0) = \int_{\mathbb{R}} \varphi(x) e^{0}\, dx$$
$$= \int \langle \varphi(x), 1 \rangle dx$$

so $\mathcal{F}(\delta) = 1$. Thus
$$L K(z) = \delta.$$
is an equivalent definition, similar to as with the heat Kernel.

Def 27.5: The operator $u \longmapsto K * u$ is called the
1) Green's operator, or 2) parametrix. It is a special case of a convolution operator.

Convolution operators are a key aspect of the study of PDEs. See Math 173/205.

27.ii: Young's Inequality for Convolutions

One fundamental result is

Thm 27.6 (Young's Convolution) $p, q, r \geq 1$.
Suppose $\frac{1}{p} + \frac{1}{q} = 1 + \frac{1}{r}$. Then $\exists C$ such that $f, g \in L^p(\mathbb{R}), L^q(\mathbb{R})$
$$\| f * g \|_r \leq C \|f\|_p \|g\|_q.$$
Hence $f * g \in L^r(\mathbb{R})$. The same holds on $\mathbb{R}^n$.

Proof:
$$f * g = \int_{\mathbb{R}} f(x-y) g(y) dy$$
$$|f * g| \leq \int_{\mathbb{R}} |f(x-y)||g(y)| dy$$
$$\leq \int_{\mathbb{R}} |f(x-y)|^{1 + p/r - p/r} |g(y)|^{1 + q/r - q/r} dy$$
$$\leq \int_{\mathbb{R}} \left(|f(x-y)|^p |g(y)|^q\right)^{1/r} |f(x-y)|^{1-p/r} |g(y)|^{1-q/r}$$
$$\leq \left(\int_{\mathbb{R}} |f(x-y)|^p |g(y)|^q\right)^{1/r} \left\| |f(x-y)|^{\frac{r-p}{r}} \right\|_{\frac{pr}{r-p}} \left\| |g(y)|^{\frac{r-q}{r}} \right\|_{\frac{qr}{r-q}}$$

Here, we used if $\frac{1}{p^*}+\frac{1}{q^*}+\frac{1}{r^*}=1$

$$\int fgh \leq \|f\|_p \|g\|_q \|h\|_r$$

multi-Hölder, (induction) applied for $r^*=r$, $p^*=\frac{pr}{r-p}$, $q^*=\frac{qr}{r-q}$.

$$\leq \left(\int |f|^p |g|^q\right)^{1/r} \|f\|_p^{\frac{r-p}{r}} \|g\|_q^{\frac{r-q}{r}}$$

translation invariance of Lebesgue!

$$\|f*g\|_r^r = \int_{\mathbb{R}} |f*g|^r \, dx$$

$$\leq \int_{\mathbb{R}} \left[\int_{\mathbb{R}} |f(x-y)|^p |g(y)|^q dy\right] \|f\|_p^{r-p} \|g\|_q^{r-q} dx$$

$$\leq \|f\|_p^{r-p} \|g\|_q^{r-q} \int_{\mathbb{R}}\int_{\mathbb{R}} |g(y)|^q |f(x-y)|^p dx\, dy \quad \text{(Fubini)}$$

$$\leq \|f\|_p^{r-p} \|g\|_q^{r-q} \int_{\mathbb{R}} g(y) \|f\|_p^p = \|f\|_p^{r-p+p} \|g\|_q^{r-q+q} \quad \square.$$

<u>Ex 27.7</u> : On $\mathbb{R}^n$ $n \geq 3$

- $K(x-y)$ for Δ is $\frac{1}{|x-y|^{n-2}}$ so $\int |K(z)| \, dV = \int \frac{1}{r^{n-2}} r^{n-1} dr$

$\Rightarrow$ the Green's function of Δ just barely fails to be L^1!

- $K(x-y)$ for $\Delta + \beta^2 = \frac{e^{-\beta|x-y|}}{|x-y|^{n-2}}$ same (but only at 0).

<u>Rem 27.8</u> : This is part of why studying integral kernels is difficult, cannot just apply Young.

27.iii) The Fourier transform of L^p

Recall $\mathcal{F}: L^2 \to L^2$ is an isometry.

Question 27.9: What is the Fourier transform of L^p for $1 \le p < \infty$?

Ex 27.10 $\mathcal{F}(1) = \delta$ and $1 \in L^\infty$ but $\delta \notin L^p$ for any p.

Thm 27.11 (Hausdorff-Young) If $1 \le p \le 2$, $\frac{1}{p} + \frac{1}{q} = 1$, then

$$\|\hat{f}\|_{L^q} \le C \|f\|_{L^p}$$

Corollary 27.12: For $1 \le p \le 2$,

$$\mathcal{F}: L^p(\mathbb{R}) \longrightarrow L^q(\mathbb{R})$$

is continuous and injective. (It is not surjective) this is extremely difficult.

Proof: Suppose that $q = 2k$ for $k \in \mathbb{N}$, $p = \frac{2k}{2k-1} = 2, \frac{4}{3}, \frac{6}{5}, \ldots$

$$\|\hat{f}\|_{L^q} = \|\hat{f}\|_{L^{2k}} = \|(\hat{f})^k\|_{L^2}^{1/k}$$

$$= \|\hat{f} \cdot \hat{f} \cdot \ldots \|_{L^2}^{1/k}$$

$$= \|f * f * f * \ldots\|_{L^2}^{1/k}$$

If $k = 2$. $1 + \frac{1}{2} = \frac{3}{2} = 2\left(\frac{3}{4}\right) = 2\left(\frac{1}{4/3}\right)$:

$$\le \|f\|_{L^p} \|f\|_{L^p}^{1/2}$$

For general k, iterated Young convolution does same □